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Some sufficient conditions for hereditarily finitely based varieties of semigroups

G. POLLÁK

Introduction

The proofs of most theorems saying that one or another variety is hereditarily finitely based are very similar to each other (in so far as syntactic proofs are concerned). The general scheme of such proofs has been described in [6] (see also Theorem 2.5 of the present paper); however, this description does not help much more in future proofs as finger-posts do in alpinism. The essential difficulty usually lies in proving that the objects and relations in the scheme are what they ought to be (sometimes it is not even easy to construct them). We think therefore that every unification which renders possible to claim that a more or less broad class of varieties is h.f.b. is of interest.

In the present paper we give sufficient conditions of the following two types: a) if J is a fully invariant ideal of the countably generated free semigroup F, and a certain quasi-ordered set (connected with $F\setminus J$) is well-quasi-ordered then the variety $SG(J)$ defined by all identities $u=v$, $u, v\in J$ is h.f.b.; b) if V is a variety, $M\subset F$ is a standard form for elements of J (i.e. every $w\in J$ equals to some $w^*\in M$), and M itself, as well as the "process of standardization" are subject to certain conditions, then every variety in the lattice interval $[V, V\cap SG(J)]$ is finitely based over V. Furthermore, we show that certain concrete subsets of F satisfy these conditions. As an application, we find all h.f.b. identities in one of the four classes of "candidates" to such identities (see [3]; class (d)). This result accomplishes, in a certain sense, the investigations concerning such identities; namely, classes (c) and (d), as well as balanced h.f.b. equations are completely described now (see [1], [4], [6]), homotypical and some other equations of class (b) are settled in [5], and it looks likely that presently known *syntactic* methods cannot help us much further.

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Part I. General sufficient conditions

1. Preliminary. We need rather a lot of not generally known concepts and of notations running through this paper; so we have collected most of them here.

By the free semigroup F we always mean the countably generated free semigroup $F(X)$ on the set of generators $X=\{x_1, x_2, \dots\}$. As we need also the free monoid F^0, we shall call the elements of F *terms*, the elements of F^0 *words* (i.e. a word can be empty, a term cannot). The coincidence of words will be denoted by $u\equiv v$; the formula $u=v$ is an identity which holds in some subvariety V of the variety of all semigroups SG. The empty word, as well as the empty set, we denote by $\emptyset$; this will not lead to confusion.

The set of all variables (letters) which occur in u will be denoted by $X(u)$. Furthermore, $|u|$ denotes the length of u, and $|u|_i$ the number of occurrences of x_i in u. The words $u_{(l)}$, $u^{(l)}$ are the prefix and the suffix of length l of the word u (of length $\geqq l$), resp.:

$$u \equiv u_{(l)}u' \equiv u''u^{(l)}, \quad |u| \geqq l, \quad |u_{(l)}| = |u^{(l)}| = l. \tag{1.1}$$

A third kind of denoting equality is defined by

$$u =! \, u_1u_2 \overset{\text{def}}{\Leftrightarrow} u \equiv u_1u_2 \quad \text{and} \quad X(u_1)\cap X(u_2) = \emptyset. \tag{1.2}$$

Note that $u=!\,u_1u_2$ implies $|X(u)|=|X(u_1)|+|X(u_2)|$. If a word has no decomposition of the form (1.2), it is said to be *irreducible*. Every word has a unique irreducible decomposition:

$$u =!! \prod_{i=1}^{r} u_i \overset{\text{def}}{\Leftrightarrow} u =! \prod_{i=1}^{r} u_i, \quad u_i \text{ irreducible for } i = 1, \dots, r. \tag{1.3}$$

We call the components u_i the *irreducible factors* of u. The word u is said to be *semiirreducible* if $|u_i|>1$ for every i (in particular, $\emptyset$ is semiirreducible). The decomposition

$$u =! \, w_0 \prod_{j=1}^{s} x_{c(i)}w_i, \quad w_i \text{ semiirreducible} \tag{1.4}$$

will be called the *semiirreducible factorization* of u, and $w_0, \dots, w_s$ its *semiirreducible factors*. A word is said to be *simple* if its semiirreducible factors are empty (i.e. if no letter occurs in it more than once). Besides (1.4), we shall also make use of the *reduced semiirreducible factorization*

(1.5)

$$u =! \, w_0 \prod_{i=1}^{s} a_i\tilde{w}_i, \quad \tilde{w}_i \text{ semiirreducible, } a_i \text{ simple, } \tilde{w}_i,\ a_i \not\equiv \emptyset \text{ for } i = 1, \dots, s-1.$$

Clearly, both (1.4) and (1.5) are unique.

By $F_{(n)}$ we denote the set of all words having only semiirreducible factors of length $\leqq n$. In particular, $F_{(0)}=F_{(1)}$ consists of all simple words. Obviously, $F_{(n)}\subset F_{(n+1)}$ for every $n\geqq 1$, and $F^0\setminus F_{(n)}=J_{(n)}$ is a fully invariant ideal in F. Let $M\subset F^0$. Set

$$(1.6)\qquad M^{[m]}=\{u\colon\ u\equiv a_0\prod_{i=1}^{t}u_ia_i,\ |a_0\dots a_t|\leqq m,\ \tilde u=!\ \tilde u_1\dots\tilde u_t\in M\},$$

$$(1.7)\qquad M^{(m)}=\{u\colon\ u\in M^{[m]},\ X(a_0\dots a_t)\cap X(\tilde u)=\emptyset\}.$$

It is easy to see that

Lemma 1.1. *$F_{(n)}^{[m]}\subseteqq F_{(n)}^{(m(n+1))}$ if $n\geqq 1$.*

Indeed, if $u\in F_{(n)}^{[m]}$, then at most m factors of the factorization (1.4) of u can contain some element of $X(a_0\dots a_t)$ (as $|X(a_0\dots a_t)|\leqq m$), and every such factor is of length $\leqq n$. Put

$$u\equiv a_0'\prod_{i=1}^{t'}u_i'a_i'$$

where every a_i' is equal to a factor of (1.4) which contains a letter of $a_0\dots a_t$. Clearly,

$$|a_0'\dots a_{t'}'|\leqq (n+1)m,\quad\text{and}\quad X(a_0'\dots a_{t'}')\cap X(u_1'\dots u_{t'}')=\emptyset.$$

Two words u, u' are said to be of the same type if there is an automorphism $\alpha\in\operatorname{Aut}F^0$ which maps u into u': $u\alpha=u'$. The set of all words of the same type (an orbit of $\operatorname{Aut}F^0$) is called a *type*. E.g. X and $0=\{\emptyset\}$ are types. The type of u will be denoted by $T(u)$. If u is irreducible, simple etc., the same is said about $T(u)$.

An endomorphism $\varphi\in\operatorname{End}F^0$ is said to be *disjoint* if $X(x_i\varphi)\cap X(x_j\varphi)=\emptyset$ provided $i\neq j$. The endomorphism φ is *finite* if $|x_i\varphi|>1$ for at most finitely many x_i's. The number

$$\gamma(\varphi)=\sum_{i=1}(|x_i\varphi|-1)$$

is called the *growth* of φ. The set of all disjoint endomorphisms will be denoted by Dend F^0, that of all finite disjoint endomorphisms by Fde F^0.

The proof of the following facts is straightforward (see also [4]).

Lemma 1.2. *If $u\notin X$ is (semi)irreducible and $\varphi\in$ Dend F^0, then $u\varphi$ is (semi)-irreducible.*

Lemma 1.3. *If $u=!\prod_{i=1}^{m}w_i$ and $\varphi\in$ Dend F^0 then $u\varphi=!\prod_{i=1}^{m}w_i\varphi$.*

Let $r(u)$ $(=r)$, $s(u)(=s)$ be the number of factors in (1.3) and (1.4), respectively.

Lemma 1.4. *If $\varphi\in$ Fde F^0 then $r(u)\leqq r(u\varphi)\leqq r(u)+\gamma(\varphi)$, $s(u)\leqq s(u\varphi)\leqq$ $\leqq s(u)+\gamma(\varphi)$, and the image of an irreducible factor $u_i\notin X$ of u is an irreducible factor of $u\varphi$.*

We have to deal with several order and quasi-order relations. The most important order relation will be the lexicographical order of words defined by

$$u <_{\text{lex}} v \overset{\text{def}}{\Leftrightarrow} v \equiv uv', \quad v' \in F \quad \text{or} \quad u \equiv wx_i u', \quad v \equiv wx_j v', \quad i<j.$$

The lexicographical order is not a well-order on F^0, however, it is a well-order on $F^0 \setminus F^n$ (the set of words of length $<n$).

Let V be a variety of semigroups, and $F(V)$ the free semigroup in V on the infinite set of generators $\{x_1\nu, x_2\nu, \ldots\}$, where $\nu\colon F \to F(V)$ is the canonical homomorphism. A fully invariant ideal J is said to be a *V-ideal* if J is the full inverse image of $J\nu$. In particular,

$$J(V) = \{u \in F\colon \text{there is a } v \in F \text{ such that } v \not\equiv u, \ V \models u=v\}$$

is a V-ideal.

Let σ be a set of identities. By $V(\sigma)$ we denote the subvariety of V consisting of those algebras which satisfy σ. If J is a fully invariant ideal then

$$V(J) = V(\tau) \quad \text{where} \quad \tau = \{u = v\colon \ u, v \in J\}$$

(i.e. $V(J)$ is generated by the algebra $F(V)/J$).

Following Petrich, we term an identity $u=v$ *homotypical* if $X(u)=X(v)$ and *heterotypical* else. The ideal

$$J_0(V) = \{u \in F\colon \ \text{there is a } v \in F \text{ such that } X(u) \neq X(v), \ V \models u=v\}$$

is a V-ideal, too. Obviously, $J_0(V) \subseteqq J(V)$, and $J_0(V)\nu$ is the kernel of $F(V)$.

Two systems of identities σ_1, σ_2 are said to be *V-equivalent* if

$$V \models \sigma_1 \Leftrightarrow \sigma_2.$$

Similarly, a system σ is *V-finite, V-independent* etc., if it is V-equivalent to a finite system, not V-equivalent to any proper subsystem of itself etc. Furthermore, $V' (\subseteqq V)$ is said to be *finitely based over* V if $V' = V(\sigma)$, σ finite (or, equivalently, V-finite). The interval $[V', V)$ of the lattice of varieties is finitely based if every element of $[V', V)$ is finitely based over V. If V is finitely based, too, then we say that $[V', V]$ is finitely based.

We say that the system of identities σ (or, also, the identity $u=v$ in the case $\sigma = \{u=v\}$) is hereditarily finitely based (h.f.b. for short) if the variety $SG(\sigma)$ is, where SG is the variety of all semigroups (which, by definition, means that every subvariety of $SG(\sigma)$ is finitely based).

Let J be a fully invariant ideal in F. A subset $M \subseteqq F$ is termed a *standard form for V in J* (or, for short, *M is standard for V in J*) if for every $u \in J$ there is a $u^* \in M$ such that $V \models u = u^*$ (neither uniqueness nor the existence of an algorithm for finding u^* is demanded). Clearly, if M is a standard form, then so is every $M' \supseteqq M$. Moreover, if J is a V-ideal, $u \in J$ and $u = u^*$ imply $u^* \in J$. Thus, $J \cap M$ is also a standard form for V in J. However, sometimes it is more convenient to work with larger standard forms. If $J=F$ we simply say that M is a standard form for V. Usually we state our theorems for standard forms in an arbitrary J because the more elegant special case of standard forms for V does not suffice in the applications.

Finally, if $\sigma=\{u_s=v_s\}$ then by $\bar{\sigma}$ we denote the system $\sigma\cup\{v_s=u_s\}$.

2. Well-quasi-orders and h.f.b. varieties. A quasi-order relation $\prec$ is said to be a *well-quasi-order* (*wqo* for short) if there are neither infinite (strictly) descending $\prec$-chains nor infinite $\prec$-antichains or, equivalently, if every infinite sequence contains an infinite (not necessarily strictly) ascending subsequence ([2]). The following quasi-orderings on subsets $M\subseteq F^0$ will occur:

$$u\lhd u' \quad \text{iff} \quad u'\equiv u_1\cdot u\varphi\cdot u_2 \quad \text{for some} \quad \varphi\in\operatorname{End}F^0, \quad u_1,u_2\in F^0,$$

$$u\lhd_V u' \quad \text{iff} \quad V\models u'=u_1\cdot u\varphi\cdot u_2 \quad \text{for some} \quad \varphi\in\operatorname{End}F^0, \quad u_1,u_2\in F^0,$$

$$u\lhd_h u' \quad \text{iff} \quad u'\equiv u\varphi \quad \text{for some} \quad \varphi\in\operatorname{End}F^0,$$

$$u\ll u' \quad \text{iff} \quad u'\equiv u\varphi \quad \text{for some} \quad \varphi\in\operatorname{Fde}F^0.$$

Note that in the last case it is sufficient to find a $\varphi'\in\operatorname{Dend}F^0$ such that $u'\equiv u\varphi'$: this can be always modified so as to obtain a $\varphi\in\operatorname{Fde}F^0$.

Let, furthermore, $\mathbf{P}$ denote the set of positive integers and Σ the symmetric group on $\mathbf{P}$. For every $k\in\mathbf{P}$ we define two quasi-orders, one on $M\times\Sigma$ and one on $M^2\times\Sigma$ as follows:

$$(u,\pi)\ll_k(u',\pi') \quad \text{iff} \quad \exists\varphi\in\operatorname{Fde}F^0(u\varphi\equiv u',\ x_{i\pi}\varphi\equiv x_{i\pi'} \text{ for } 1\leqq i\leqq k),$$

$$(u,v;\pi)\ll_k^2(u',v';\pi') \quad \text{iff} \quad \exists\varphi\in\operatorname{Fde}F^0\quad(u\varphi\equiv u',\ x_{i\pi}\varphi\equiv x_{i\pi'}$$
$$\text{for } 1\leqq i\leqq k,\ v\varphi\in M).$$

The remark made above about φ is valid here, too. Also, it is worth noting that in the definition of $\ll_k^2$ the word v' does not play any role, and that $\ll$ is a wqo if $\ll_k$ is.

In proving varieties to be h.f.b., often it is crucial to know that one or the other of these quasi-orders is a wqo for some standard form M. As for the second one, this is even indispensable:

Lemma 2.1. *If V is h.f.b., then $\lhd_V$ is a wqo on F^0.*

Proof. Obviously, no infinite descending $\lhd_V$-chain can exist. If $u_1,u_2,\dots$ were an infinite $\lhd_V$-antichain, then consider the system $\sigma=\{u_{2k-1}=u_{2k}\colon k=1,2,\dots\}$. We are going to show that $V(\sigma)$ is not finitely based, moreover, if $\sigma_k=\sigma\setminus\{u_{2k-1}=u_{2k}\}$ then $V(\sigma_k)\not\models u_{2k-1}=u_{2k}$. Indeed, suppose there exist $(u_{2k-1}\equiv)v_1,\dots,v_l(\equiv u_{2k})$ such that $v_i\equiv v_i'\cdot w_i\varphi_i\cdot v_i''$, $v_{i+1}\equiv v_i'\cdot w_i'\varphi_i\cdot v_i''$ for $i=1,\dots,l-1$, where $\varphi_i\in\operatorname{End}F$, $v_i',v_i''\in F^0$ and either $V\models w_i=w_i'$ or $w_i=w_i'\in\bar{\sigma}_k$. If it is always the first instance that prevails then $V\models u_{2k-1}=u_{2k}$, which is not the case. Let i be the minimal index such that $V \not\models v_i = v_{i+1}$, and, say, $w_i \equiv u_{2r-1}$, $w_i' \equiv u_{2r}$, $r \neq k$. Then $V \models u_{2k-1} = v_i \equiv \equiv v' \cdot u_{2r-1}\varphi_i \cdot v_i''$, contrary to the assumption.

Clearly, if $\ll_k$ is a wqo (for some fixed M) then $\ll_l$ for $0 \leqq l < k$ is a wqo, too. However, already the condition that $\ll_1$ is a wqo on some $M \times \Sigma$ is rather restrictive as the following lemma shows.

Lemma 2.2. *If $\ll_1$ is a wqo on $M \times \Sigma$ then there is a natural number p such that $|u|_i \leqq p$ for every $u \in M$ and $i \in \mathbf{P}$.*

Proof. Put $p(u) = \max |u|_i$, and suppose $p(u_1) < p(u_2) < \ldots$, $|u_j|_{k(j)} = p(u_j)$. Then we obtain an infinite $\ll_1$-antichain (u_j, π_j), $j = 1, 2, \ldots$ by putting $\pi_j = (1\ k(j))$: if φ is disjoint and $x_{k(j)}\varphi \equiv x_{k(i)}$, $j \neq i$, then $|u_j\varphi|_{k(i)} = p(j) \neq p(i)$ whence $u_j\varphi \not\equiv u_i$.

Let $G_{n,k} = \{(a_1, \ldots, a_r) \in (F^0)^r \colon |a_1 \ldots a_r| \leqq n,\ |X(a_1 \ldots a_r)| \leqq k\}$. The following proposition enables a more flexible handling of $\ll_k$.

Proposition 2.3. *$\langle M \times \Sigma, \ll_k \rangle$ is wqo iff $M \times G_{n,k}$ is wqo for every n under*

$$(u;\ a_1, \ldots, a_r) \prec (u';\ a_1', \ldots, a_{r'}') \ \textit{iff}\ \ r = r' \ \textit{and}$$
$$\exists \varphi \in \mathrm{Fde}\, F^0 (u\varphi \equiv u',\ a_i\varphi \equiv a_i' \textit{ for } i = 1, \ldots, r). \tag{2.1}$$

Proof. The sufficiency is obvious because $(u, \pi) \ll_k (u', \pi')$ is equivalent to $(u; x_{1\pi}, \ldots, x_{k\pi}) \prec (u'; x_{1\pi'}, \ldots, x_{k\pi'})$. Now let $\ll_k$ be a wqo, and let $x_{s(1)}, \ldots, x_{s(q)}$ be the variables of $a_1 \ldots a_r$ (in the order of their first occurrence). Set

$$\pi = \begin{pmatrix} 1 & \ldots & q \\ s(1) & \ldots & s(q) \end{pmatrix},$$

and put $(u; a_1, \ldots, a_r) \prec_0 (u'; a_1', \ldots, a_{r'}')$ iff $r = r'$, $T(a_1 \ldots a_r) = T(a_1' \ldots a_{r'}')$, $T(a_i) = T(a_i')$ for $i = 1, \ldots, r$, $(u, \pi) \ll_q (u', \pi')$. Now $\prec_0$ is a wqo because $q \leqq k$ and r, $T(a_1 \ldots a_r)$, $T(a_i)$ can take only a finite number of different "values". Furthermore, $(u; a_1, \ldots, a_r) \prec_0 (u'; a_1', \ldots, a_{r'}')$ implies $(u; a_1, \ldots, a_r) \prec (u'; a_1', \ldots, ar')$ because the very endomorphism φ which satisfies $u\varphi \equiv u'$, $x_{s(j)}\varphi \equiv x_{s'(j)}$ maps a_i on a_i'. Hence $\prec$ is a wqo.

The following proposition shows that the conditions that $\ll_k$ is a wqo for different numbers k are not very far from each other.

Proposition 2.4. *If $M \subset F^0$, and there is a natural number q such that $uv \in M$ implies $|X(u) \cap X(v)| \leqq q$, furthermore, $\ll_{2q+1}$ is a wqo on M, then $\ll_k$ is a wqo on M for every k.*

Proof. For $k \leqq 2q+1$ (in particular, for $k = 1$) the assertion is obvious. So suppose it holds for some k. For $(u, \pi) \in M \times \Sigma$ consider the decomposition

$$u \equiv u_0 \prod_{i=1}^{r} x_{t_i\pi} u_i,\ t_i \leqq k,\ X(u_i) \cap \{x_{1\pi}, \ldots, x_{k\pi}\} = \emptyset,$$

and put

$$\hat{u} \equiv \prod_{i=1}^{r} x_{t_i\pi}, \quad \bar{u} \equiv u_0 \dots u_r, \quad \bar{u}_i \equiv u_0 \dots u_{i-1}, \quad \bar{\bar{u}}_i \equiv u_i \dots u_r,$$

$$X_i(u) = (X(\bar{u}_i) \cap X(\bar{\bar{u}}_i)) \setminus \{x_{(k+1)\pi}\}, \quad \bar{X}(u) = \bigcup_{i=1}^{r} X_i(u), \quad \bar{X}(u_i) = \bar{X}(u) \cap X(u_i).$$

It is easy to see that $r \leqq kp$ where p is the bound from Lemma 2.2, $|\bar{X}(u)| \leqq rq$, and $\bar{X}(u_i) \subseteq X_i(u) \cup X_{i+1}(u)$, whence $q_i = |\bar{X}(u_i)| \leqq 2q$. Choose permutations $\pi_0, \dots, \pi_r$ such that $1\pi_i = (k+1)\pi$, $x_{2\pi_i}, \dots, x_{(q_i+1)\pi_i}$ are the different elements of $\bar{X}(u_i)$ (e.g. in the order of their first occurrence in u_i), and define

$$(u, \pi) \prec (u', \pi') \quad \text{iff} \quad (u, \pi) \ll_k (u', \pi'),$$

(2.2)

$$(u_i, \pi_i) \ll_{2q+1} (u_i', \pi_i') \quad \text{for} \quad i = 1, \dots, r, \quad \text{and} \quad T\Big(\prod_{i=0}^{r} \prod_{t=1}^{q_i+1} x_{t\pi_i}\Big) = T\Big(\prod_{i=0}^{r} \prod_{t=1}^{q_i'+1} x_{t\pi_i'}\Big).$$

As $(u, \pi) \ll_k (u', \pi')$ implies $T(\hat{u}) = T(\hat{u}')$, r is the same for (u, π) and for (u', π'), and the definition makes sense. Furthermore, (2.2) implies that $q_i = q_i'$ for $0 \leqq i \leqq r$, because $1\pi_i = (k+1)\pi$, $1\pi_i' = (k+1)\pi'$ for every i; in particular, $x_{(k+1)\pi}$ and $x_{(k+1)\pi'}$ are the first letters of the corresponding products, and this fixes the length of the inner products.

From the assumptions it follows that $\prec$ is a wqo on $M \times \Sigma$, so it is sufficient to show that $\prec$ is weaker than $\ll_{k+1}$, i.e. if (2.2) holds then there is a $\varphi \in$ Fde F^0 such that $u\varphi \equiv u'$, $x_{t\pi}\varphi \equiv x_{t\pi'}$ for $t \leqq k+1$. By (2.2), there are disjoint endomorphisms ψ, $\varphi_0, \dots, \varphi_r$ such that

$$u\psi \equiv u', \quad x_{t\pi}\psi \equiv x_{t\pi'} \quad \text{for} \quad 1 \leqq t \leqq k,$$

$$u_i\varphi_i \equiv u_i', \quad x_{t\pi_i}\varphi_i \equiv x_{t\pi_i'} \quad \text{for} \quad 1 \leqq t \leqq q_i+1, \quad i = 1, \dots, r.$$

Put

$$x_s\varphi \equiv \begin{cases} x_s\varphi_i & \text{if} \quad x_s \in X(u_i), \\ x_s\psi & \text{if} \quad x_s \notin X(\bar{u}). \end{cases}$$

Then φ is well-defined: if $x_s \in X(u_i) \cap X(u_j)$, $i<j$, then $s = i\pi_i = t'\pi_j$ for some $t \leqq q_i + 1$, $t' \leqq q_j + 1$, however, then, the third condition in (2.2) guarantees that $t\pi_i' = t'\pi_j'$, i.e. $x_s\varphi_i \equiv x_s\varphi_j$. Also, it is not difficult to see that φ is disjoint. Furthermore, $\hat{u}\psi \equiv \hat{u}'$ whence $\hat{u}' \equiv \Pi x_{t_i\pi'}$ (in general, the sequence $t_1, \dots, t_r$ depends on (u, π)). Thus, $u\varphi \equiv u'$, $x_{t\pi}\varphi \equiv x_{t\pi'}$ for $t = 1, \dots, k+1$.

In [6], the generally used syntactic method of proving varieties to be h.f.b. is formulated in Proposition 2.1. Here we give a slightly different version. Let $M \subset F^0$, V a variety, and J a fully invariant ideal of F. We say that M is a *good standard form for V in J* (or a *good standard form for V* if $J = F$) if there exist a linear order relation

$<$ on M and a quasi-order relation $\prec$ on $M^*=\{(u,v)\colon u>v\}\subset M^2$ such that the following conditions are fulfilled:

C) For every $u\in J$ there is a $u^*\in M$ such that $V\models u=u^*$;

O) $(M,<)$ is a well-ordered set;

Q) $(M^*,\prec)$ is a wqo set;

A) If $(u,v),(u',v')\in M^*$, $(u,v)\prec(u',v')$ then there is a $w\in M$ such that $w<u'$ and $V\models(u=v\Rightarrow u'=w)$.

If one replaces F^r by J in the first part of the proof of Proposition 2.1 in [6], one obtains

Proposition 2.5. *If there is a good standard form for V in J then the interval $[V(J),V)$ is finitely based.*

It is easy to see that if M is a good standard form for V in J then it is a good standard form in the minimal V-ideal J_V which contains J. Thus, $J_V\cap M$ is a standard form for V in J (even in J_V).

In order to obtain a sufficient condition for V to be h.f.b., we need the following

Proposition 2.6. *Let V be a variety and J a V-ideal. If the interval $[V(J),V)$ is finitely based and $V(J)$ is h.f. b. over V then every subvariety of V is finitely based over V.*

The proof is based on

Lemma 2.7. *Let V be a variety, J a fully invariant ideal in F, and $V'=V(J)$. Let, furthermore, σ be a system of identities and $u\in F$ such that $V(\sigma)\not\models u=v$ for any $v\in J$. Then $V'(\sigma)\models u=u'$ implies ($u'\notin J$ and) $V(\sigma)\models u=u'$.*

Proof. Let $(u\equiv)\ v_1,v_2,\dots,v_l\ (\equiv u')$ be a sequence of terms such that $v_j\equiv\equiv v_j'\cdot w_j\varphi_j\cdot v_j''$, $v_{j+1}\equiv v_j'\cdot w_j'\varphi_j\cdot w_j''$ for $j=1,\dots,l-1$, where $v_j',v_j''\in F^0$, $\varphi_j\in\operatorname{End}F$, and either $w_j,w_j'\in J$ or $V\models w_j=w_j'$ or $(w_j=w_j')\in\bar\sigma$. Suppose l_0 is the least index for which $w_{l_0},w_{l_0}'\in J$. Then $V(\sigma)\models u=v_{l_0}\in J$, contrary to the assumption. Hence $v_j\notin J$ $(j=1,\dots,l)$ and $v_1,\dots,v_l$ yields a proof of $u=u'$ in $V(\sigma)$.

Proof of Proposition 2.6. A system of identities σ can be supposed, without loss of generality, to consist of three parts $\sigma_J=\{(u=u')\in\sigma\colon u,u'\in J\}$, $\sigma_J'==\{(u=u')\in\sigma\colon u,u'\notin J\}$ and $\sigma_0=\{(u=u')\in\sigma\colon u\in J,u'\notin J\}$. Using Lemma 2.1, we can replace all but finitely many members of σ_0 by identities of type σ_J: if $(u=u')$, $(v=v')\in\sigma_0$ and $u'\lhd_{V(J)}v'$ then $V(J)\models v'=u_1\cdot u'\varphi\cdot u_2$ $(u_1,u_2\in F^0,\varphi\in\operatorname{End}F)$; however, $v'\notin J$ whence $V\models v'=u_1\cdot u'\varphi\cdot u_2$ and therefore $\{u=u',v=v'\}$ is V-equivalent to $\{u=u',v=u_1\cdot u'\varphi\cdot u_2\}$. Thus, one can assume that σ_0 is finite. The same holds for σ_J' because $V(J)(\sigma_J')$ is finitely based by assumption, whence σ_J' is $V(J)$-equivalent to some finite system σ^*, but then Lemma 2.7 implies also $V(\sigma_J')=V(\sigma^*)$.

Finally, $V(\sigma_J)\in[V(J), V]$ and therefore is finitely based over V. This completes the proof.

Putting together Propositions 2.5 and 2.6, we obtain a condition which can be used in proving varieties to be h.f.b. In these applications, the following lemma will be referred to several times.

Lemma 2.8. *Let V be a variety, J a fully invariant ideal in F, and M a standard form for V in J subject to the condition*

(i) *for every $u\in J$ there is a $u^*\in M$ such that $V\models u=u^*$ and $|X(u^*)|\leqq|X(u)|$.*

If $u, v\in J$, $X(u)\nsubseteq X(v)$ then for every $\varphi\in\mathrm{Dend}\,F$ with $|X(u\varphi)|\neq 1$ there is a $w\in M$ such that $|X(w)|<|X(u\varphi)|$ and $V\models u=v\Rightarrow u=w$.

Proof. Choose an arbitrary $y\in X(v\varphi)\cap X(u\varphi)$ if $X(v\varphi)\cap X(u\varphi)\neq\emptyset$ and put $y\equiv x_1$ else. Define $\psi\in\mathrm{End}\,F$ by

$$x\psi\equiv\begin{cases}x\varphi & \text{if } x\in X(u),\\ y & \text{if } x\notin X(u).\end{cases}$$

Then $u\psi\equiv u\varphi$, and $X(v\psi)=X(v\varphi)\cap X(u\varphi)$ if $X(v\varphi)\cap X(u\varphi)\neq\emptyset$, $X(v\psi)=\{y\}$ else. Now $X(v)\cap X(u)\subset X(u)$; hence $X(v\varphi)\cap X(u\varphi)\subset X(u\varphi)$ because φ is disjoint. Thus, $|X(v\psi)|<|X(u\varphi)|$, and, in virtue of (i), the term $w\equiv(v\psi)^*$ meets the requirements.

Now we give a sufficient condition, which may seem rather sophisticated at first glance, however, can be applied to reasonable classes of varieties. By $\langle u, v\rangle$ we denote the greatest common prefix of u and v, i.e. the longest subword w such that $u\equiv w\bar{u}$, $v\equiv w\bar{v}$.

Theorem 2.9. *Let V be a variety of semigroups, J a fully invariant ideal in F, $M_i\subseteqq F^0$ for $i=1, \dots, l$, and let*

$$M=\{u\equiv u_1\dots u_l\colon u_i\in M_i,\ \text{and } u\equiv\tilde{u}\bar{u}\Rightarrow X(\tilde{u})\cap X(\bar{u})<g\}$$

be a standard form for V in J with some natural number g. Suppose, moreover, that the following conditions are fulfilled with some natural numbers n and $k\geqq n+(l-1)g+2$; condition (i) *from* Lemma 2.7, *furthermore,*

(ii) *$\langle M_i\times\Sigma, \ll_k\rangle$ is a wqo set for $i=1, \dots, l$;*

(iii) *if $v\equiv\tilde{v}\bar{v}\equiv v_1\dots v_l\in M$, $\tilde{v}\equiv v_1\dots v_{j-1}\tilde{v}_j$, $v_j\equiv\tilde{v}_j\bar{v}_j$, $\varphi\in\mathrm{Fde}\,F^0$, and $v\varphi$ is a prefix of some $v'\equiv v'_1\dots v'_l\in M$ such that $v'_i\equiv v_i$ for $i<j$, furthermore, $|\tilde{v}^{(n)}\varphi|=n$, then* (i) *is satisfied for $v\varphi$ with some $(v\varphi)^*$ s. th. either $(v\varphi)^*_i\equiv\equiv v_i\varphi$ for $i=1, \dots, j-1$, $|\langle v_j\varphi, (v\varphi)^*_j\rangle|\geqq|\tilde{v}_j\varphi|-n$ or, for some $h\leqq j$, $(v\varphi)^*_i\equiv\equiv v_i\varphi$ for $i<h$, $(v\varphi)^*_h$ is a proper prefix of $v_h\varphi$ (of $\tilde{v}_j\varphi$, if $h=j$).*

Then $[V(J), V)$ is finitely based.

Proof. Fix a factorization $u \equiv u_1 \dots u_l$, $u_i \in M_i$, for every $u \in M$. Define $<$ on M by

$$v < u \quad \text{iff either} \quad |X(v)| < |X(u)|,$$
$$\text{or} \quad |X(v)| = |X(u)|, \quad v_i \equiv u_i \quad \text{for} \quad i = 1, \dots, j-1, \quad v_j <_{\mathrm{lex}} u_j.$$

Furthermore, for $(u, v) \in M^*$ set $u_i \equiv v_i$ if $i<j$, $u_j \not\equiv v_j$, and

$$u \equiv w x_e \bar{u}, \quad w \equiv u_1 \dots u_{j-1} \tilde{v}_j, \quad \tilde{v}_j \equiv \langle u_j, v_j \rangle,$$

$$v_j \equiv \begin{cases} \tilde{v}_j & \text{or} \\ \tilde{v}_j x_d \hat{v} \bar{v}_j, & |\hat{v}| = n \quad \text{or} \quad |\hat{v}| < n, \quad \bar{v}_j = \emptyset, \end{cases} \qquad v \equiv \begin{cases} w\bar{v} & \text{or} \\ w x_d \hat{v} \bar{v}. \end{cases}$$

If $v_j \equiv \tilde{v}_j$, we put $d=e$, $\hat{v} \equiv \emptyset$. Let u_0 denote the product of all different variables of $\bigcup_{i<h} (X(u_i) \cap X(u_h)) = \bigcup_{i=1}^{l-1} (X(u_1 \dots u_i) \cap X(u_{i+1} \dots u_l))$ (which shows that $|u_0| \leqq (l-1)g$) in the order of their first occurrence in u, say. Suppose that $x_{c(1)}, \dots, x_{c(r)}$ is the sequence of all different letters of $a \equiv x_d x_e (w x_d \hat{v})^{(n)} u_0$; clearly, $r \leqq |a| \leqq (l-1)g + n+2=k$. Choose $\pi \in \Sigma$ such that $t\pi = c(t)$ for $t \leqq r$, and for (u, v), $(u', v') \in M^*$ put

$$(u, v) \prec (u', v') \quad \text{iff} \quad (u_i, \pi) \ll_k (u_i', \pi') \quad \text{for} \quad i=1, \dots, l,$$
$$r = r', \quad j = j', \quad |w|_e = |w'|_{e'}, \quad T(x_d x_e \hat{v}) = T(x_{d'} x_{e'} \hat{v}'),$$
$$|X(u)| = |X(v)| \Leftrightarrow |X(u')| = |X(v')|, \quad w \equiv v \Leftrightarrow w' \equiv v', \quad \tilde{v}_j \equiv v_j \Leftrightarrow \tilde{v}_j' \equiv v_j'$$

(letters with$'$ denote objects which belong to (u', v')).

Lemma 2.2 implies that $(M, <)$ is a well-ordered set, because $|u| \leqq p|X(u)|$ with some constant p. Furthermore, as $r, j, |w|_e$ and $|x_d x_e \hat{v}|$ are bounded, and the equivalences decompose M^* in eight $\prec$-independent classes, the qo-set $(M^*, \prec)$ is wqo in virtue of (ii). Thus, it remains to prove that (A) is satisfied.

If $(u, v) \prec (u', v')$ then there are disjoint endomorphisms $\varphi_1, \dots, \varphi_l$ such that $u_i \varphi_i \equiv u_i'$, $x_{c(t)} \varphi_i \equiv x_{c'(t)}$ for $t=1, \dots, r$ and $i=1, \dots, l$; in particular, φ_i and φ_h coincide on $X(u_i) \cap X(u_h)$. Hence the endomorphism φ given by

$$x_s \varphi \equiv \begin{cases} x_s \varphi_i & \text{if} \quad x_s \in X(u_i), \\ x_s \varphi_1 & \text{if} \quad x_s \notin X(u) \end{cases}$$

is well-defined and disjoint. We have $u_i \varphi \equiv u_i'$, $u\varphi \equiv u'$. Hence $V \models u' = (v\varphi)^*$, and all we have to show is $(v\varphi)^* < u'$. Clearly, we can suppose $|x_s \varphi| = 1$ for $x_s \notin X(u)$; hence, by (i),

$$|X(u')| - |X((v\varphi)^*)| \geqq |X(u\varphi)| - |X(v\varphi)| \geqq |X(u)| - |X(v)| \geqq 0.$$

This proves the assertion for the case $|X(v)| < |X(u)|$ (in virtue of Lemma 2.8, even for $X(v) \neq X(u)$ if $|X(u')| \neq 1$). So let $|X(v)| = |X(u)|$, $|X(u')| = |X((v\varphi)^*)|$. Then we have also $|X(u')| = |X(v')|$. Next note that $|w|_e = |w'|_{e'}$ and $x_e \varphi \equiv x_{e'}$ (as $e = 2\pi$, $e' = 2\pi'$) guarantee $w\varphi \equiv w'$. If, moreover, $\tilde{v}_j \equiv v_j$ then $\tilde{v}_j' \equiv v_j'$, i.e. $v_i \varphi \equiv v_i'$ for the first j

components of v'. Hence, according to (iii), either $(v\varphi)_1^* \equiv v_1', \ldots, (v\varphi)_{h-1}^* \equiv v_{h-1}'$, $(v\varphi)_h^*$ is a proper prefix of v_h' for some $h \leqq j$ or $(v\varphi)_i^* \equiv v_i'$ for $i=1, \ldots, j$. In the first case $(v\varphi)^* < v' < u'$, in the second one $(v\varphi)^* < u'$, too, because $v_j' <_{\text{lex}} u_j'$. If, on the other hand, $v_j \equiv \tilde{v}_j x_d \hat{v} \bar{v}_j$, then $v_j' \equiv \tilde{v}_j' x_{d'} \hat{v}' \bar{v}_j'$, $d<e$, $d'<e'$ (because $v<u$, $v'<u'$), and $T(x_d x_e \hat{v}) = T(x_{d'} x_{e'} \hat{v}')$ implies, in particular, $|\hat{v}| = |\hat{v}'|$. Now either $|\hat{v}|<n$, then we have $v_j' \equiv \tilde{v}_j' x_{d'} \hat{v}' \equiv (\tilde{v}_j x_d \hat{v})\varphi \equiv v_j \varphi$, and the same argument as for the case $\tilde{v}_j \equiv v_j$ prevails (only $v_j' < u_j'$ follows now from $d'<e'$); or $|\hat{v}|=n$, and, again by (iii), either there is an $h \leqq j$ such that $(v\varphi)_i^* \equiv v_i \varphi \equiv v_i'$ for $i<h$, $(v\varphi)_h^*$ is a proper prefix of v_h' (of $\tilde{v}_j' x_{d'} \hat{v}'$ if $h=j$) whence $(v\varphi)^* < v'$, or $(v\varphi)_i^* \equiv v_i'$ for $i=1, \ldots, j-1$, $|\langle v_j', (v\varphi)_j^* \rangle| \geqq |\tilde{v}_j' x_{d'} \hat{v}'|$, i.e. $(v\varphi)_j^* \equiv v_j' x_{d'} \bar{w}_j <_{\text{lex}} u_j'$, which yields the proof for this case, the last one.

In the special case $l=1$ Condition (iii) reads somewhat simpler:

Corollary 2.10. *Let V be a variety of semigroups, J a fully invariant ideal in F, and $M \subset F^0$ standard for V in J. Suppose that $\langle M, \ll_k \rangle$ is a wqo set for some $k \geqq 2$,* (i) *holds, and for some $n \leqq k-2$ we have*

(iii′) *if $v \equiv \tilde{v}\bar{v} \in M$, $\varphi \in \mathrm{Fde}\, F^0$, $\tilde{v}\varphi$ is a prefix of some $v' \in M$, and $|\tilde{v}^{(n)}\varphi| = n$, then* (i) *satisfied for some standard form $(v\varphi)^*$ of $v\varphi$ such that either $(v\varphi)^*$ is a prefix of $v\varphi$ or $|\langle v\varphi, (v\varphi)^* \rangle| \geqq |v\varphi| - n$.*

Then $[V(J), V)$ is finitely based.

Remark 1. If V is homotypical (i) is fulfilled.

Remark 2. It is not difficult to distil from the proof that (iii) can be weakened in the following manner. Instead of $v \in M$ we consider pairs $(v, \varrho) \in M \times \Sigma$, and we require (iii) only for those $\varphi \in \mathrm{Fde}\, F^0$ such that $|\chi_i \varphi| = 1$ for $1 \leqq i \leqq k-n-2$. This enables us to dispose not only of the elements of $X(x_d x_e (w x_d \hat{v})^{(n)} u_0)$ but of some more variables, too, provided k is sufficiently large. It is precisely in this form that Theorem 2.9 will be applied at the end of the paper.

The next two theorems are devoted to special cases where the conditions can be weakened.

Theorem 2.11. *Let I be a fully invariant ideal of F. If $\langle (F \setminus I) \times \Sigma, \ll_2 \rangle$ is a wqo set then $SG(I)$ is h.f.b.*

Proof. Choose an $a \in I$, and set $M = (F \setminus I) \cup \{a\}$. Define $v<u$ iff either $v \equiv a$, $u \in F \setminus I$ or $v, u \in F \setminus I$ and $v <_{\text{lex}} u$. By Lemma 2.2, $<$ is a well-order.

For $(u, v) \in M^*$ put

$$u \equiv w x_e \bar{u}, \quad v \equiv \begin{cases} w & \text{or} \\ w x_d \bar{v}, & d \neq e, \end{cases} \quad \pi = \begin{pmatrix} 1 & 2 \ldots \\ d & e \ldots \end{pmatrix}$$

if $v \not\equiv w$. Define

$$(u, v) \prec (u', v') \quad \text{iff either} \quad u \ll u', \ v \equiv a, \quad \text{or} \quad u \ll u', \quad v \equiv w,$$

$$\text{or} \quad (u, \pi) \ll_2 (u', \pi'), \quad v' \not\equiv a, w.$$

Clearly, $\prec$ is a wqo relation on M^*. Note that $u, u' \not\equiv a$ and if $v \equiv w$ then also $v \not\equiv a$.

Anyway, there is a $\varphi \in \mathrm{Fde}\, F$ such that $u' \equiv u\varphi$. If $|X(v)| < |X(u)|$ then we can suppose that $|X(v\varphi)| < |X(u\varphi)|$ whence $v\varphi < u\varphi$. Now let $|X(v)| = |X(u)|$. If $u \ll u'$, $v \equiv a$ then $u' \equiv u\varphi = (v\varphi)^* \equiv a < u'$. If $u \ll u'$, $v \equiv w$, then $u' \equiv u\varphi \equiv w\varphi \cdot (x_e \bar{u})\varphi > w\varphi \equiv v\varphi$. If $v, v' \notin \{a, w\}$, $(u, \pi) \ll_2 (u', \pi')$, then $u' \equiv u\varphi$, $x_{d'} \equiv x_d \varphi$, $x_{e'} \equiv x_e \varphi$, $d' < e'$. Hence $u\varphi \equiv w\varphi \cdot x_{e'} \cdot \bar{u}\varphi$, $v\varphi \equiv w\varphi \cdot x_{d'} \cdot \bar{v}\varphi$, and either $v\varphi \in I$, $(v\varphi)^* \equiv a < u'$, or $v\varphi \in F \setminus I$, $v\varphi <_{\mathrm{lex}} u'$ whence $v\varphi < u'$. This completes the proof.

Theorem 2.12. *If V, J, M are as in* Corollary 2.10, (i) *holds, and $\langle M^2 \times \Sigma, \ll_2^2 \rangle$ is wqo, then $[V(J), V)$ is finitely based.*

Proof. Define $<$ by

$$v < u \quad \text{iff either} \quad |X(v)| < |X(u)| \quad \text{or} \quad |X(v)| = |X(u)|, \quad v <_{\mathrm{lex}} u.$$

For $(u, v) \in M^*$ set $w \equiv \langle u, v \rangle$, $u \equiv w x_e \bar{u}$, $v \equiv w$ or $v \equiv w x_d \bar{v}$, $d < e$, and put $d = 1 + \max \{i \colon x_i \in X(u) \cup X(v)\}$, $\bar{v} \equiv \emptyset$ if $v \equiv w$. Let

$$\pi = \begin{pmatrix} 1 & 2 \dots \\ d & e \dots \end{pmatrix},$$

and define

$$(u, v) \prec (u', v') \quad \text{iff} \quad (u, v; \pi) \ll_2^2 (u', v'; \pi'), \quad |w|_e = |w'|_{e'},$$

$$X(v) = X(u) \Leftrightarrow X(v') = X(u'), \quad v \equiv w \Leftrightarrow v' \equiv w'.$$

Obviously, $<$ is a well-order on M and $\prec$ is a wqo on M^*. If $X(v) \neq X(u)$ and $|X(u')| \neq 1$ then (A) follows from Lemma 2.8. If $|X(u')| = 1$ then $|X(u)| = 1$ by $u \ll u'$ and $|X(v')| = |X(v)| = 1$ by $v' < u'$, $v < u$. If, besides, $X(v) \neq X(u)$ then $X(v') \neq X(u')$; hence $u \equiv x_e^m, v \equiv x_d^n, u' \equiv x_{e'}^{m'}$, $v' \equiv x_{d'}^{n'}$, $d < e$, $d' < e'$, $m | m'$, and putting $x_e \varphi \equiv x_{e'}^{m'/m}$, $x_d \varphi \equiv x_{d'}$, we have $u' \equiv u\varphi$, $v\varphi = (v\varphi)^* < u'$. Finally, let $X(v) = X(u)$. Then $X(v') = X(u')$, and there is a $\varphi \in \mathrm{Fde}\, F^0$ such that $u\varphi \equiv u'$, $x_d \varphi \equiv x_{d'}$, $x_e \varphi \equiv x_{e'}$, $v\varphi \in M$. Now $|w|_e = |w'|_{e'}$ implies $w\varphi \equiv w'$, and either $v \equiv w$, $v\varphi \equiv w\varphi \equiv w' \equiv v' < u'$ or $v \equiv w x_d \bar{v}$, $v\varphi \equiv w' x_{d'} \cdot \bar{v}\varphi < u'$ because $d' < e'$. This completes the proof.

Remark. Of course, $\langle M^2 \times \Sigma, \ll_2^2 \rangle$ is wqo if and only if $\langle M^* \times \Sigma, \ll_2^2 \rangle$ is. Hence we could have defined in advance and replaced M^2 by M^* in the text of the theorem. This will be our way in the next section (Lemma 3.7).

3. Standard forms and h.f.b. varieties. In this section we show some (comparatively large) particular subsets of F^0 to be good standard forms whenever they are standard and part of the conditions (i)—(iii) (or (iii′)) is satisfied.

Theorem 3.1. *If* $F^{(p)}_{(n)}$ *is standard for* V *in* J *and* (i), (iii′) *hold* (*with some* $n \geqq 0$), *then* $[V(J), V)$ *is finitely based.*

Proof. By Corollary 2.10, it is sufficient to show that $\langle F^{(p)}_{(n)} \times \Sigma, \ll_k \rangle$ is a wqo set for every k. The proof will be accomplished through a succession of lemmas.

Lemma 3.2. $F_{(n)}$ *is wqo under* $\ll$.

Proof. Denote by T_n the set of all semiireducible types of length $\leqq n$, and define the quasi-order relation $\prec$ on $\mathbf{P} \times \mathsf{T}_n$ by

$$(k, T) \prec (k', T') \quad \text{iff} \quad k < k', \quad T = T'.$$

As T_n is finite, $\mathbf{P} \times \mathsf{T}_n$ is a wqo set under $\prec$, and, according to [2], so is the set V of vectors $(\alpha_1, \ldots, \alpha_l)$, $\alpha_i \in \mathbf{P} \times \mathsf{T}_n$, of arbitrary length under the relation

$$(\alpha_1, \ldots, \alpha_l) \prec (\beta_1, \ldots, \beta_m) \quad \text{if there is a sequence} \quad i(1) < \ldots < i(l) = m$$
$$\text{such that} \quad \alpha_j \prec \beta_{i(j)}$$

(the additional condition $i(l) = m$ accepted here obviously does not change the situation). Assign to $u \in F_{(n)}$ the vector $\mathbf{a}(u) = (\alpha_1, \ldots, \alpha_{\bar{s}})$ where $\alpha_i = (|a_i|, T(\tilde{w}_i))$ (see (1.5)), and put

$$u \prec u' \quad \text{iff} \quad \mathbf{a}(u) \prec \mathbf{a}(u') \quad \text{and} \quad T(\tilde{w}_0) = T(\tilde{w}_0').$$

This relation is a wqo, too, because $F^0_{(n)} \to V \times \mathsf{T}_n$ $(u \mapsto (\mathbf{a}(u), T(\tilde{w}_0)))$ is a strict homomorphism of $(F^0_{(n)}, \prec)$ onto a wqo set. On the other hand, if $u \prec u'$ then we can construct a $\varphi \in \mathrm{Fde}\, F^0$ such that $u\varphi \equiv u'$: there exist endomorphisms $\varphi_j \in \mathrm{Fde}\, F^0$, $j = 0, \ldots, \bar{s}$, with $(a_j \tilde{w}_j)\varphi_j \equiv a'_{i(j)} \tilde{w}'_{i(j)}$ (where $i(0) = 0$, $a_0 \equiv a_0' \equiv \emptyset$). Denote the first letter of a_j by $x_{r(j)}$, and put

$$x_t \varphi \equiv \begin{cases} x_t \varphi_j & \text{if} \quad x_t \in X(a_j w_j), \quad t \neq r(j), \\ \left(\prod\limits_{k = i(j-1)+1}^{i(j)-1} a_k' \tilde{w}_k' \right) \cdot x_{r(j)} \varphi_j & \text{if} \quad t = r(j), \\ x_{t+N} & \text{if} \quad x_t \notin X(u), \end{cases}$$

where $N = \max\limits_{x_i \in X(u')} i$. It is easy to see that φ fits for our aim.

Lemma 3.3. *If* M *is closed for subwords,* $\ll$ *is a wqo on* M, *and the length of the irreducible elements of* M *is bounded then* $\ll_k$ *is a wqo on* $M \times \Sigma$ *for every* $k \geqq 0$.

Proof. Let $(u, \pi) \in M \times \Sigma$ and suppose that $u_{i(1)}, \ldots, u_{i(l)}$ $(i(1) < \ldots < i(l))$ are those irreducible factors of u which contain some $x_{s\pi}$, $s \leqq k$. Put $u_\pi \equiv u_{i(1)} \ldots$

$\dots u_{i(l)}$ and suppose that $x_{s\pi}$ occurs in u_π for the first time at the m_s-th place (i.e. $u_\pi \equiv \bar{u}_\pi x_{s\pi} \bar{\bar{u}}_\pi$, $x_{s\pi} \notin X(\bar{u}_\pi)$, $|\bar{u}_\pi| = m_s - 1$); if $x_{s\pi} \notin X(u)$ put $m_s = \infty$. Clearly, $u =! \, v_0 \prod_{j=1}^{l} u_{i(j)} v_j$. Assign to (u, π) the vector

$$\mathbf{b}(u, \pi) = \langle m_1, \dots, m_k; l; T(u_{i(1)}), \dots, T(u_{i(l)}); v_0, \dots, v_l \rangle$$

and set

$$\mathbf{b}(u, \pi) \prec \mathbf{b}(u', \pi') \quad \text{iff} \quad m_s = m_s', \quad l = l', \quad T(u_{i(r)}) = T(u'_{i(r)}),$$

$$v_j \ll v_j' \quad \text{for} \quad 1 \leqq s \leqq k, \quad 1 \leqq r \leqq l, \quad 0 \leqq j \leqq l.$$

Then $(u, \pi) \prec (u', \pi') \Leftrightarrow \mathbf{b}(u, \pi) \prec \mathbf{b}(u', \pi')$ defines a wqo on $M \times \Sigma$. Indeed, $(u, \pi) \mapsto \mathbf{b}(u, \pi)$ is then a strict homomorphism between quasi-ordered sets, and the set of the vectors $\mathbf{b}$ is wqo under $\prec$ because if N is an upper bound of $|u|$ for irreducible $u \in M$ then $l \leqq k$, $m_s \leqq |u_\pi| \leqq kN$, and $T(u_{i(r)})$ can take also only a finite number of different values and the assertion follows from Lemma 3.2. Furthermore, if $(u, \pi) \prec (u', \pi')$ then we can define $\varphi \in \mathrm{Fde}\, F^0$ so that $v_j \varphi \equiv v_j'$, $u_{i(r)} \varphi \equiv u'_{i(r)}$, and then $|u_\pi| = |u'_{\pi'}|$, $m_s = m_s'$ guarantees also $x_{s\pi} \varphi \equiv x_{s\pi'}$ for $s = 1, \dots, k$, i.e. $(u, k) \ll_k (u', \pi')$.

As an immediate consequence of Lemmas 3.2 and 3.3 we get

Corollary 3.4. *$F_{(n)} \times \Sigma$ is wqo under $\ll_k$ for every $k > 0$.*

Note, however, that Lemma 3.3 is only seemingly more general than Corollary 3.4 because it is not difficult to see that if the conditions of the lemma are fulfilled then $M \subseteqq F_{(n)}$ for some n.

Finally we prove

Lemma 3.5. *$F_{(n)}^{(p)} \times \Sigma$ is a wqo set under $\ll_k$ for every $k > 0$.*

Proof. For $(u, \pi) \in F_{(n)}^{(p)} \times \Sigma$ let $s_1 < \dots < s_l \leqq k$ be those indices for which $x_{s_i\pi} \in X(a_0 \dots a_t)$ (see (1.6)), and suppose that $x_{s_i\pi}$ occurs in $a_0 \dots a_t$ for the first time at the m_i-th place. Assign to (u, π) the vector

$$\mathbf{c}(u, \pi) = \mathbf{c}\langle l, t; s_1, \dots, s_l; m_1, \dots, m_l; T(a_0), \dots, T(a_t), T(a_0 \dots a_t); \tilde{u}_1, \dots, \tilde{u}_t \rangle$$

and put

$$\mathbf{c}(u, \pi) \prec \mathbf{c}(u', \pi') \quad \text{if the first} \quad 2l + t + 4 \quad \text{components of both}$$

$$\text{vectors coincide and} \quad (\tilde{u}_j, \pi) \ll_k (\tilde{u}_j', \pi') \quad \text{for} \quad j = 1, \dots, t.$$

Define $(u, \pi) \prec (u', \pi') \Leftrightarrow \mathbf{c}(u, \pi) \prec \mathbf{c}(u', \pi')$. As in the proof of Lemma 3.4, we can see that $F_{(n)}^{(p)} \times \Sigma$ is a wqo set under $\prec$. Furthermore, if $(u, \pi) \prec (u', \pi')$ then we can define $\varphi_0 \in \mathrm{Fde}\, F^0$ such that $(a_0 \dots a_t) \varphi_0 \equiv a_0' \dots a_t'$ which guarantees also $x_{s_i\pi} \varphi_0 \equiv x_{s_i\pi'}$, $a_r \varphi_0 = a_r'$ for $i = 1, \dots, l$; $r = 0, \dots, t$, and $\varphi_j \in \mathrm{Fde}\, F^0$ such that $\tilde{u}_j \varphi_j \equiv \tilde{u}_j'$, $x_{s\pi} \varphi_j \equiv x_{s\pi'}$. Putting together $\varphi_0, \dots, \varphi_j$ (which is possible in virtue of (1.6)—(1.7)), we obtain a $\varphi \in \mathrm{Fde}\, F^0$ with $u\varphi \equiv u'$, $x_{s\pi}\varphi \equiv x_{s\pi'}$ for $s=1, \dots, k$. This proves the lemma and also the theorem.

In some special cases one can omit condition (iii′). We give here one theorem of this kind.

Theorem 3.6. *If* $F_{(1)}^{(p)}$ *is standard for* V *in* J *and* (i) *holds then* $[V(J), V)$ *is finitely based.*

In virtue of Theorem 2.12, it suffices to prove

Lemma 3.7. $F_{(1)}^{(p)*} \times \Sigma$ *is a wqo set under* $\ll_k^2$ *for every* $k \geqq 1$.

Proof. Let $(u, v; \pi) \in F_{(1)}^{(p)*} \times \Sigma$, $v \equiv b_0 \prod_{i=1}^{r} \tilde{v}_i b_i$ be the decomposition of v indicated in (1.6). By Lemma 3.5 and Proposition 2.3, $F_{(1)}^{(p)} \times G_{2p+k,k}$ is wqo under $\prec$ defined in (2.1). Put

$$(u, v;\ \pi) \prec (u', v';\ \pi') \overset{\text{def}}{\Leftrightarrow}$$

$$\overset{\text{def}}{\Leftrightarrow} (u;\ x_{1\pi}, \dots, x_{k\pi}, a_0, \dots, a_t, b_0, \dots, b_r) \prec (u';\ x_{1\pi'}, \dots, x_{k\pi'}, a_0', \dots, a_{t'}', b_0', \dots, b_{r'}'),$$

$$t = t', \quad r = r', \quad |a_i| = |a_i'|, \quad |b_j| = |b_j'|.$$

Clearly $\prec$ is a wqo on $F_{(1)}^{(p)*} \times \Sigma$, and there is a $\varphi \in \mathrm{Fde}\, F^0$ such that $u\varphi \equiv u'$, $a_i\varphi \equiv \equiv a_i'$, $b_j\varphi \equiv b_j'$. Moreover, we can suppose that $x_l\varphi$ is simple for every $x_l \in X$ (for $x_l \in X(u)$ this holds automatically, as $\tilde{u}\varphi \equiv \tilde{u}'$, and $|x_l\varphi| = 1$ for $x_l \in X(a_1 \dots a_r)$). Thus, $v\varphi \equiv b_0' \prod_{i=1}^{r} \tilde{v}_i\varphi \cdot b_i' \in F_{(1)}^{(p)}$, because $\tilde{v}_i\varphi \in F_{(1)}$, $\left|\prod_{i=0}^{r} b_i'\right| \leqq p$. This proves the lemma.

Theorem 3.8. *Let* $M_i \subseteqq F_{(n)}^{(p)}$ *for* $i=1, \dots, l$. *If*

$$M = \{u \equiv u_1 \dots u_l\colon\ u_i \in M_i,\ |X(u_i) \cap X(u_j)| \leqq q \text{ for } i \neq j\}$$

is standard for V *in* J *and* (i), (iii) *hold (with some* $n \geqq 0$), *then* $[V(J), V)$ *is finitely based.*

Proof. It is easy to see that if $\tilde{u}\bar{u} \in M$ then $|X(\tilde{u}) \cap X(\bar{u})| \leqq (n+p)/2 + l^2q/4$. Now the assertion follows from Lemma 3.5 and Theorem 2.9.

We mention two more special cases.

Theorem 3.9. *If* $F_{(n)}$ *is a standard form for* V *and* (i) *holds then* V *is h.f.b.*

Proof. The theorem becomes a special case of Theorem 3.1 (with $J=F$) if we show that (iii′) holds (with $n+1$ instead of n). So let $v \equiv \tilde{v}\bar{v} \in F_{(n)}$, $v \equiv v_0 \prod_{i=1}^{s} x_{c(i)} v_i$

its semiirreducible factorization, $|v_i|\leqq n$,

$$\tilde{v} =! v_0 \prod_{i=1}^{l-1} x_{c(i)} v_i \cdot x_{c(l)} \tilde{v}_l, \quad \bar{v} = !\bar{v} \prod_{i=l+1}^{s} x_{c(i)} v_i, \quad \tilde{v}_l \bar{v}_l \equiv \tilde{v},$$

and $\varphi \in$ F de F^0,

$$\tilde{v}\varphi =! v_0' \prod_{j=1}^{r} x_{d(j)} v_j',$$

$|v_j'|\leqq n$ (i.e. $\tilde{v}\varphi \in F_{(n)}$), $|v^{(n+1)}\varphi| = n+1$. As $|\tilde{v}_l|\leqq n$, we have $|x_{c(l)}\varphi|=1$. Furthermore, by Lemma 1.3, $v\varphi =! \, w \cdot x_{c(l)}\varphi \cdot \bar{w}$. Hence $w \cdot x_{c(l)}\varphi \cdot \bar{w}^* \in F_{(n)}$, since it can be obviously achieved that $X(w \cdot x_{c(l)}\varphi) \cap X(\bar{w}^*) = \emptyset$. This proves the assertion, as $|\bar{v}\varphi| = |\bar{v}| < n$.

Proposition 3.10. *Let* $J = F_{(n)}^{(p)}$. *The variety* $SG(J)$ *is h.f.b.*

Proof. Follows from Theorem 2.11 and Lemma 3.5.

Part II. Application: a class of h.f.b. identities

The aim of this part is to prove the following

Theorem A. *A non-balanced identity of the form*

$$(*) \qquad u \equiv x_1 \dots x_i x_{i+1} x_i x_{i+2} \dots x_n = x_{1\pi}^{e(1)} \dots x_{m\pi}^{e(m)} \equiv v \quad (\pi \in \Sigma, e(j) \leqq 2)$$

is h.f.b. if and only if v is not of the form

$$(**) \qquad v \equiv x_1 \dots x_{i-1} x_{i\pi}^2 x_{(i+1)\pi}^2 x_{i+2} \dots x_n \quad (\pi = (i\; i+1) \text{ or identical, } n > 2).$$

The assertion will be broken up into several propositions. From know on V denotes $SG(*)$.

4. Two special cases. To start with, we settle the negative part of the assertion. Of course, here one cannot utilize the results of Part I.

Proposition 4.1. *The identity*

$$(\tau) \qquad x_1 \dots x_i x_{i+1} x_i x_{i+2} \dots x_n = x_1 \dots x_{i-1} x_i^2 x_{i+1}^2 x_{i+2} \dots x_n$$

is h.f.b. iff $n=2$.

Proof. The fact that $xyx = x^2y^2$ is h.f.b. has been proved in [5]. So let $n>2$ and, say, $i>1$. Consider the identity

$$(\sigma_m) \qquad tyztx_1^2 \dots x_{2m}^2 = tyztx_1^2 \dots x_{2m-2}^2 x_{2m}^2 x_{2m-1}^2$$

and the infinite systems $(\sigma)=\{\sigma_m\colon m=1, \dots\}$, $(\sigma^k)=\{\sigma_m\colon m\neq k\}$. We claim that the system $\sigma\cup\{\tau\}$ is independent. Clearly τ does not follow from σ, so we have to prove that $\sigma^k\cup\{\tau\}\nvdash\sigma_k$. To see this, we show that if $\sigma^k\cup\{\tau\}\vdash tyztx_1^2\dots x_{2k}^2=w$ then

$$(4.1)\qquad \begin{gathered} w=!(tyzt)\cdot\Big(\prod_{j=1}^{l}p_j\Big),\quad p_j\in T(xyx)\cup T(x^2),\\ p_l\equiv x_{2k}^2\quad\text{or}\quad p_l\equiv x_{2k-1}x_{2k}x_{2k-1},\quad X\Big(\prod_{j=1}^{l}p_j\Big)=\{x_1,\dots,x_{2k}\}, \end{gathered}$$

whence $w\not\equiv tyztx_1^2\dots x_{2k-2}^2x_{2k}^2x_{2k-1}^2$. Indeed, let $w_0(\equiv tyzt\,x_1^2\dots x_{2k}^2)$, $w_1,\dots,w_r(\equiv w)$ be a sequence of terms such that, for every $s=1,\dots,r$, there exist $w_s', w_s''\in F^0$, $\varphi_s\in\operatorname{End}F$, and $(u_s=v_s)\in\overline{\sigma^k\cup\{\tau\}}$ which satisfy $w_{s-1}\equiv w_s'\cdot u_s\varphi_s\cdot w_s''$, $w_s\equiv w_s'\cdot v_s\varphi_s\cdot w_s''$, Suppose, furthermore, that w_{s-1} is of the form (4.1) for some $s\leqq r$ (this certainly is the case for $s=1$). First let $u_s=v_s\in\sigma^k$; by symmetry, we can assume that $u_s\equiv tyztx_1^2\dots x_{2m}^2$. Sure enough, $m<k$ because $tyztx_1^2\dots x_{2m}^2\not\mathrel{\triangleleft} w_{s-1}$ for $m>k$. Moreover, the only subword of w_{s-1} which is an endomorphic image of $tyzt$ is $tyzt$ itself, and the only subwords which are squares are those of the form x_j^2. Hence $u_s\varphi\in T(u_s)$ and

$$w_{s-1}=!\,u_s\varphi_s\prod_{j=2m+1}^{l}p_j\equiv tyztx_{1\pi}^2\dots x_{(2m)\pi}^2\prod_{j=2m+1}^{l}p_j,\quad w_s=!\,v_s\varphi_s\cdot\prod_{j=2m+1}^{l}p_j,$$

$\pi\in\Sigma_{2m}$, $l\geqq 2m+2$, whence also w_s is of the form (4.1). If

$$u_s\equiv x_1\dots x_{i-1}x_i^2x_{i+1}^2x_{i+2}\dots x_n,\quad v_s\equiv x_1\dots x_ix_{i+1}x_ix_{i+2}\dots x_n$$

then $(x_i^2x_{i+1}^2)\varphi_s\in T(x^2y^2)$ by the same reason as above, i.e. $(x_i^2x_{i+1}^2)\varphi_s\equiv p_{j-1}p_j$ for some $j\leqq l$, and w_s differs from w_{s-1} only in these factors which are replaced by some $p_j'\in T(xyx)$; thus, w_s again is of the form (4.1) because if $j=l$ (which, by the way, can occur only if $i+1=n$) then $x_{2k-1}^2x_{2k}^2$ is replaced by $x_{2k-1}x_{2k}x_{2k-1}$. Finally, if $u_s\equiv x_1\dots x_ix_{i+1}x_ix_{i+2}\dots x_n$ then $(x_ix_{i+1}x_i)\varphi_s\in T(xyx)$ because the only subwords of w_{s-1} which are endomorphic images of xyx are those of the form $x_qx_rx_q$ $(\equiv p_j$ for some $j)$ and $tyzt$, but this latter one is out of consideration because if $(x_ix_{i+1}x_i)\varphi_s\in tyzt$ then $u_s\varphi\equiv u_s'tyxtu_s''$ with $u_s'\not\equiv\emptyset$ since $i>1$, which is impossible. Therefore this case is similar to the previous one.

If $i=1$ then $n>i+1$ and we can consider the identities

$$(\sigma_m')\qquad x_1^2\dots x_{2m}^2tyzt=x_2^2x_1^2x_3^2\dots x_{2m}^2tyzt$$

instead of σ_m, and dualize the above reasoning (with the only — unessential — difference that here $p_1\equiv x_1^2$ or $x_1x_2x_1$ which is not dual to $p_l\equiv x_{2k-1}x_{2k}x_{2k-1}$). This completes the proof.

Next we deal with a special case.

Lemma 4.2. *The identity*

$$(u \equiv)\, x_1 \ldots x_i x_{i+1} x_i \ldots x_n = x_{1\pi} \ldots x_{m\pi} (\equiv v) \quad (\pi \in \Sigma) \tag{4.2}$$

is h.f.b.

Proof. First suppose that the symmetrical difference $X(u) \div X(v) = \{x_i\}$. By repeated applications of (4.2), any word w of F^{n+1} can be brought to the form $w^* \equiv \equiv w^*_{(i-1)} x^{d(1)}_{c(1)} \ldots x^{d(k)}_{c(k)} w^{*(n-i-1)}$, $d(j) \leqq 2$; besides, the number of variables either does not change or decreases by 1 at every step. Furthermore, applying (4.2) twice, using the endomorphisms

$$x_j \varphi \equiv \begin{cases} x_{2i} & \text{if } j = i, \\ x_{2i+j} & \text{if } j \neq i, \end{cases} \qquad x_j \psi \equiv \begin{cases} x_{i+j} & \text{if } j \leqq i, \\ x_{2i+j} & \text{if } j > i+2, \\ x_{2i} \ldots x_{3i-1} & \text{if } j = i+1, \\ x_{3i+1} x_{2i} x_{3i+2} & \text{if } j = i+2, \end{cases}$$

respectively, we obtain

$$x_1 \ldots x_{2i-1} x_{2i}^2 x_{2i+1\pi} \ldots x_{2i+(n-1)\pi} x_{2i+n+1} \ldots x_{2n+i-1} =$$

$$= x_1 \ldots x_{2i-1} x_{2i}^2 \cdot v\varphi \cdot x_{2i+n+1} \ldots x_{2n+i+1} = x_1 \ldots x_{2i-1} x_{2i}^2 \cdot u\varphi \cdot x_{2i+n+1} \ldots x_{2n+i-1} =$$

$$= x_1 \ldots x_{2i-1} x_{2i}^2 x_{2i+1} \ldots x_{3i-1} x_{2i} x_{3i+1} x_{2i} x_{3i+2} \ldots x_{2n+i-1} =$$

$$= x_1 \ldots x_i \cdot u\psi \cdot x_{2i+n+1} \ldots x_{2n+i-1} = x_1 \ldots x_i \cdot v\psi \cdot x_{2i+n+1} \ldots x_{2n+i-1} =$$

$$= x_1 \ldots x_i w x_{2i} w' x_{2i} w'' x_{2i+n+1} \ldots x_{2n+i-1}, \quad w' \not\equiv \emptyset,$$

and a third application of (4.2) relieves us from x_{2i}. Hence $V \models w^* = w^{**} \equiv \equiv w^{**}_{(2i-1)} x_{f(1)} \ldots x_{f(r)} w^{**(2n-i-2)} \in F^{[2n+i-3]}_{(1)}$ and $X(w^{**}) \subseteq X(w^*) \subseteq X(w)$. Thus, by Lemma 1.1 and Theorem 3.6 $[V(F^{2n+i-3}), V)$ is finitely based, whence the assertion follows obviously.

If (4.2) is heterotypical, and $X(u) \div X(v) \neq \{x_i\}$, we can assume that $k\pi > n$ for some $k \leqq m$ (i.e. $x_{k\pi} \notin X(u)$). Indeed, if this is not the case, then there is a $j \leqq n$, $j \neq i$, such that $j\pi^{-1} > m$ (i.e. x_j does not occur on the right side). First let $j \neq i+1$, say $j > i+1$, and let j be maximal. Executing in (4.2) the substitution $x_j \mapsto x_j x_{n+1}$ on the one hand, and $x_{j+1} \mapsto x_{n+1}$, $x_{j+2} \mapsto x_{j+1}, \ldots, x_n \mapsto x_{n-1}$ on the other, we obtain

$$u \equiv x_1 \ldots x_i x_{i+1} x_i \ldots x_n = x_1 \ldots x_i x_{i+1} x_i \ldots (x_j x_{n+1}) x_{j+1} \ldots x_n = x_{1\varrho} \ldots x_{(n+1)\varrho}$$

where $\varrho = \pi \cdot (n+1\; n \ldots j+1)$, and x_{n+1} occurs on the right side. Furthermore, if $i+1 \notin \{1, \ldots, m\}$ then

$$x_1 \ldots x_i x_{i+1} x_i \ldots x_{n+2} = x_1 \ldots x_i x_{i+3} x_i x_{i+2} x_{i+3} \ldots x_{n+2}$$

and by means of the substitution $x_{i-1} \mapsto x_{i-1} x_i$, $x_i \mapsto x_{i+3}$, $x_{i+1} \mapsto x_i x_{i+2}$, $x_{i+t} \mapsto \mapsto x_{i+t+2}$ $(2 \leqq t \leqq n)$ we can bring about that x_{i+2} did not figure on the right side which is the previous case.

Now if $x_{k\pi} \notin X(u)$, $k \leqq m$ then (4.2) implies

$$x_1 \dots x_m = x_1 \dots x_{k-1}(x_k x_{k'}) \dots x_m$$

and every term is equal to one of length $\leqq m$ in the variety $V = SG(u = x_{1\pi} \dots x_{m\pi})$ whence V is h.f.b.

Now let (4.2) be homotypical. Next we show that (4.2) implies either

$$(4.3) \qquad x_1 \dots x_k x_{k+1} x_k \dots x_l = x_1 \dots x_l$$

for some $k \geqq i$, $l \geqq n$ or the dual of (4.3) — the latter if $\pi = (i\, i+1)$. In this case, as well as if $\pi = \iota$ (identical), the assertion is obvious. In the opposite case there is an $r \leqq n$, $r \notin \{i, i+1\}$, $r\pi \neq r$. Let e.g. $r < i$ and minimal. Then

$$(4.4) \qquad x_0 \dots x_n = x_0 \dots x_{r-1} x_{r\pi^{-1}} \dots x_{i\pi^{-1}} x_{(i+1)\pi^{-1}} x_{i\pi^{-1}} \dots x_{n\pi^{-1}} =$$

$$= x_0 \dots x_{r-2} x_r \dots (x_{r-1} x_{r\pi^{-1}}) \dots x_n.$$

Thus, (4.3) implies a permutative identity

$$x_1 \dots x_{n+2} = x_1 \dots x_{r-1} x_{r\varrho} \dots x_{s\varrho} x_{s+1} \dots x_{n+1}, \quad r\varrho \neq r, \quad s\varrho \neq s,$$

and, according to [7], for sufficiently large l we have

$$x_1 \dots x_l = x_1 \dots x_{r-1} x_{r\sigma} \dots x_{(s+l-n-1)\sigma} x_{s+l-n} \dots x_l$$

for every permutation σ of the symbols $r, \dots, s+l-n-1$ whence, in particular, (4.3) follows.

Using (4.3), an arbitrary word w can be easily transformed to the form $w = w_{(k-1)} w' w^{(l-k-1)}$, $xyx \not\dashv w'$, i.e. the irreducible factors of w' are contained in $X \cup T(x^2)$. However, (4.3) implies also

$$x_1 \dots x_k x_{k+1}^2 x_{k+2} \dots x_l = x_1 \dots x_k x_{k+1} x_k x_{k+1} \dots x_l =$$

$$= x_1 \dots x_k x_{k+1} x_k x_{k+2} \dots x_l = x_1 \dots x_l$$

whence $w = w_{(k)} w'' w^{(l-k-1)}$, $w'' \in F_{(1)}$. Thus, $w^* \equiv w_{(k)} w'' w^{(l-k-1)} \in F_{(1)}^{[l-1]} \subseteqq F_{(1)}^{(2l-2)}$, and (i) holds as (4.4) is homotypical. Hence the assertion of the lemma follows by Theorem 3.6 (here $J = F$).

5. Some auxiliary identities. From now on we can suppose, in virtue of Lemma 4.2 and the results of [6], that

$$(5.1) \qquad e(k) = 2 \quad \text{for some} \quad k \leqq m, \quad k\pi^{-1} \neq i.$$

We proceed by some identities which follow from $(*)$ and (5.1). Note that

$$(5.2) \qquad (*) \vdash x^{n+2} = x^{n+4}$$

as $(*)$ is supposed to be non-balanced. Hence $(x^{2n})v$ is an idempotent in $F(V)$. Moreover, if $(*)$ is homotypical then

$$(5.3)\qquad (*) \vdash x^{n+2} = x^{n+3}$$

and already $(x^{n+2})v$ is idempotent.

Lemma 5.1. *Suppose that* (5.1) *holds. If* (5.3) *holds, too, then*

$$(5.4)\qquad (*) \vdash x^{n+2}y^q z^{n+2} = x^{n+2}yz^{n+2} \quad \textit{for} \quad q \geqq 1.$$

If $(*)$ *is heterotypical (in particular, if* (5.3) *does not hold) then*

$$(5.5)\quad (*) \vdash x^{n+2}y^{2q+1}z^{n+2} = x^{n+2}yz^{n+2},\quad x^{n+2}y^{2q}z^{n+2} = x^{n+2}z^{n+2} \quad \textit{for} \quad q \geqq 0.$$

Proof. If (5.3) holds and $(*)$ is heterotypical then [5], Lemma 2 yields even $x^{n+2}yz^{n+2}=x^{n+2}z^{n+2}$. So let $(*)$ be homotypical, and first suppose $\pi\neq\iota$, $\pi\neq(i\, i+1)$. There is a $k\notin\{i, i+1\}$ such that $k\pi\neq k$; let e.g. $k<i$ and choose k to be minimal. Furthermore, there is an $l\neq i$ such that $e(l)=2$, as $(*)$ is non-balanced. Now put $k=s\pi$ and

$$x_t\varphi \equiv \begin{cases} x^{n+2} & \text{if} \quad t<k, \\ y & \text{if} \quad t=k, \\ z^{n+2} & \text{if} \quad t>k; \end{cases} \quad x_t\psi \equiv \begin{cases} y^{e(s)} & \text{if} \quad t=l\pi, \\ z^{n+2} & \text{else}; \end{cases}$$

$$x_t\varphi' \equiv \begin{cases} y^2 & \text{if} \quad t=k, \\ x_t\varphi & \text{else.} \end{cases}$$

We have in virtue of $(*)$

$$\begin{aligned} x^{n+2}yz^{n+2} &= x^{n+2}\cdot u\varphi\cdot z^{n+2} = x^{n+2}\cdot v\varphi\cdot z^{n+2} = \\ &= x^{n+2}z^{n+2}y^{e(s)}z^{n+2} = x^{n+2}z^{n+2}\cdot u\psi\cdot z^{n+2} = \\ &= x^{n+2}z^{n+2}\cdot v\psi\cdot z^{n+2} = x^{n+2}z^{n+2}y^{2e(s)}z^{n+2} = \\ &= x^{n+2}\cdot v\varphi'\cdot z^{n+2} = x^{n+2}\cdot u\varphi'\cdot z^{n+2} = x^{n+2}y^2z^{n+2}, \end{aligned}$$

which implies (5.4). If, on the other hand, $\pi=\iota$, then the substitution of

$$x_t\chi \equiv \begin{cases} x^{n+2} & \text{if} \quad t<l, \\ y & \text{if} \quad t=l, \\ z^{n+2} & \text{if} \quad t>l, \end{cases}$$

in $(*)$ yields (5.4) immediately.

If $(*)$ is heterotypical, we can confine ourselves, by the remark made above, to the case where (5.3) does not hold. According to [5], Lemma 2, it suffices to prove $x^{2n}y^{2n}x^{2n}=x^{2n}$. However, in our case there is a $k<m$ with $e(k)=2$, $k\pi>n$. The substitution

$$x_t\vartheta \equiv \begin{cases} y^n & \text{if } t = k\pi, \\ x^{2n} & \text{else} \end{cases}$$

in $(*)$ yields the required identity.

Define

$$m_1 = \min\{j\colon j\pi \neq j\}, \quad m_2 = \min\{j\colon e(j) = 2\};$$

if $m_1 > \min(m, n) \neq \max(m, n)$, we put $\pi = (m+1\ \ m+2)$, $m_1 = m+1$, and if $m_1 > \max(m, n)$, let $\pi = \iota$, $m_1 = \infty$.

Lemma 5.2. *If* $(*) \vdash x^{n+2} = x^{n+3}$ *and either* $m_1 < m_2$ *or* $m_2 \notin \{i, i+1\}$ *or* $m_1 = m_2 = i+1$ *then*

(5.6) $$(*) \vdash x_1 \dots x_{n-1} y^{n+2} = x_1 \dots x_{n-2} x_{n-1}^2 y^{n+2}.$$

Proof. First consider the case $m_1 < m_2$. Choose $\varphi, \psi \in \operatorname{End} F$ such that

$$x_t\varphi \equiv \begin{cases} x_t & \text{if } t < m_1 \\ x_{m_1} & \text{if } t = m_1\pi, \\ y^{n+2} & \text{else;} \end{cases} \quad x_t\psi \equiv \begin{cases} x_{m_1}^2 & \text{if } t = m_1\pi, \\ x_t\varphi & \text{else.} \end{cases}$$

In virtue of $(*)$ and Lemma 5.1 we have

$$x_1 \dots x_{m_1} y^{n+2} = v\varphi \cdot y^{n+2} = u\varphi \cdot y^{n+2} = x_1 \dots x_{m_1-1} (y^{n+2} x_{m_1})^{\varepsilon} y^{n+2} =$$
$$= x_1 \dots x_{m_1-1} (y^{n+2} x_{m_1}^2) y^{n+2} = u\psi \cdot y^{n+2} = v\psi \cdot y^{n+2} = x_1 \dots x_{m_1-1} x_{m_1}^2 y^{n+2}$$

$$\left(\varepsilon = \begin{cases} 2 & \text{if } m_1\pi = i, \\ 0 & \text{if } m_1\pi > n, \\ 1 & \text{else} \end{cases} \right).$$

Hence (5.6) follows.

Next let $m_2 \leqq \min(i-1, m_1)$, and put

$$x_t\varphi \equiv \begin{cases} x_t & \text{if } t \leqq m_2, \\ y^{n+2} & \text{if } t > m_2. \end{cases}$$

Then

$$x_1 \dots x_{m_2} y^{n+2} = u\varphi \cdot y^{n+2} =$$
$$= v\varphi \cdot y^{n+2} = \begin{cases} x_1 \dots x_{m_2-1} x_{m_2}^2 y^{n+2} & \text{if } m_2 < m_1, \\ x_1 \dots x_{m_2-1} y^{n+1} x_{m_2}^{e(s)} y^{n+2} & \text{if } m_2 = m_1 = s\pi. \end{cases}$$

In the second case we define $\psi \in \operatorname{End} F$ by

$$x_t\psi \equiv \begin{cases} x_{m_2}^2 & \text{if } t = m_2 \\ x_t & \text{else} \end{cases}$$

and obtain

$$x_1\dots x_{m_2-1}y^{n+2}x_{m_2}^{e(s)}y^{n+2}=x_1\dots x_{m_2-1}y^{n+2}x_{m_2}^{2e(s)}y^{n+2}=$$
$$=v\psi\cdot y^{n+2}=u\psi\cdot y^{n+2}=x_1\dots x_{m_2-1}x_{m_2}^2y^{n+2}.$$

In both cases (5.6) follows.

If $i+1<m_2\leqq m_1$, we get, by obvious substitutions in $(*)$,

$$x_1\dots x_{i+1}y^{n+2}=x_1\dots x_ix_{i+1}x_i\,y^{n+2}=x_1\dots x_ix_{i+1}x_ix_{i+1}y^{n+2}=\dots$$
$$\dots=x_1\dots x_ix_{i+1}x_ix_{i+1}^{n+2}y^{n+2}=x_1\dots x_ix_{i+1}^{n+2}y^{n+2}=x_1\dots x_ix_{i+1}^2y^{n+2}.$$

Finally, if $m_1=m_2=i+1$, put

$$x_t\varphi\equiv\begin{cases}x_t & \text{if } t\leqq i,\\ y^{n+2} & \text{if } t>i;\end{cases}\quad x_t\psi\equiv\begin{cases}x_t\varphi & \text{if } t\leqq i+1,\\ x_i^{n+2} & \text{if } t>i+1;\end{cases}\quad x_t\chi\equiv\begin{cases}x_i^{n+2} & \text{if } t\leqq i,\\ y^{n+2} & \text{if } t>i.\end{cases}$$

Applying $(*)$ and Lemma 5.1, we obtain

$$x_1\dots x_iy^{n+2}=v\varphi\cdot y^{n+2}=u\varphi\cdot y^{n+2}=x_1\dots x_iy^{n+2}x_iy^{n+2}=$$
$$=x_1\dots x_iy^{n+2}x_i^{n+2}y^{n+2}=u\psi\cdot x_i^{n+2}y^{n+2}=v\psi\cdot x_i^{n+2}y^{n+2}=x_1\dots x_{i-1}(x_i^{n+2}y^{n+2})^2=$$
$$=x_1\dots x_{i-1}\cdot u\chi\cdot y^{n+2}=x_1\dots x_{i-1}\cdot v\chi\cdot y^{n+2}=x_1\dots x_{i-1}x_i^{n+2}y^{n+2}=x_1\dots x_{i-1}x_i^2y^{n+2}.$$

This completes the proof.

Lemma 5.3. *If $(*)$ is heterotypical and either $m_1\leqq m_2$ or $m_2\neq i$, then*

$$(*)\vdash x_1\dots x_{n-1}y^{n+2}=x_1\dots x_{n-1}x_n^2y^{n+2}. \tag{5.7}$$

Proof. If $m_1\leqq m_2$ then set

$$x_t\varphi\equiv\begin{cases}x_t & \text{if } t<m_1,\\ y^{2n} & \text{if } t\geqq m_1,\end{cases}\quad x_t\psi\equiv\begin{cases}x_{m_1}^2 & \text{if } t=m_1\pi,\\ x_t\varphi & \text{else.}\end{cases}$$

By $(*)$ and Lemma 5.1, we have (taking in account that $e(m_1)\neq 0$ as $m_1\leqq m_2\leqq m$)

$$x_1\dots x_{m_1-1}y^{n+2}=v\varphi\cdot y^{n+2}=u\varphi\cdot y^{n+2}=u\psi\cdot y^{2n}x_{m_1}^{2/e(m_1)}y^{n+2}=$$
$$=v\psi\cdot y^{n+2}=x_1\dots x_{m_1-1}x_{m_1}^2y^{n+2},$$

which implies (5.7).

If $i\neq m_2<m_1$ then $(*)\vdash x^{n+1}=x^{n+2}$ and in consequence of Lemma 5.2 and Lemma 5.1 we have even

$$x_1\dots x_{n-1}y^{n+1}=x_1\dots x_{n-2}x_{n-1}^2y^{n+1}=x_1\dots x_{n-2}x_{n-1}^{n+1}y^{n+1}=$$
$$=x_1\dots x_{n-2}x_{n-1}^{n+1}x_n^{n+1}y^{n+1}=x_1\dots x_{n-1}x_n^{n+1}y^{n+1}=x_1\dots x_ny^{n+1}.$$

Clearly, both (5.6) and (5.7) imply

$$x_1\dots x_{n-1}y^{n+2}=x_1\dots x_{n-2}x_{n-1}^3y^{n+2}. \tag{5.8}$$

Furthermore, it is easy to see that

(5.9) $\quad (5.6) \vdash x_1 \dots x_n y^{n+2} = x_1 \dots x_{n-2} x_{n-1}^2 x_n y^{n+2} = x_1 \dots x_{n-2} x_{n-1}^{n+2} x_n y^{n+2},$

(5.10) $\quad (5.7) \vdash x_1 \dots x_n y^{n+2} = x_1 \dots x_{n-1} x_{n+1}^2 x_n y^{n+2},$

(5.11) $\quad (5.8) \vdash x_1 \dots x_n y^{n+2} = x_1 \dots x_{n-2} x_{n-1}^3 x_n y^{n+2} = x_1 \dots x_{n-2} x_{n-1}^{2n+1} x_n y^{n+2},$

(5.12) $\quad (5.6) \wedge (5.7) \vdash x_1 \dots x_n y^{n+2} = x_1 \dots x_{n-1} y^{n+2} = x_1 \dots x_{n-2} x_n y^{n+2}.$

Remark. The only cases when neither the conditions of the Lemmas 5.2, 5.3, nor those of their duals are fulfilled, are given by $(**)$ and

(5.13) $$v \equiv x_1 \dots x_{i-1} x_i^2 x_{i+1} \dots x_n.$$

Lemma 5.4. $(*) \vdash (x^{2n} y^{2n})^2 = x^{2n} y^{2n}$.

Proof. If $(*)$ is heterotypical the assertion follows from Lemma 5.1. So let $(*)$ be homotypical. We indicate the substitutions in $(*)$ which yield the required identity.

If $\pi = \iota$, substitute

$$x_t \equiv \begin{cases} x^{2n} & \text{if } t \leqq i, \\ y^{2n} & \text{if } t > i. \end{cases}$$

If $k\pi \neq k$ for some $k \notin \{i, i+1\}$ (suppose e.g. $k < i$), choose k to be minimal and set

$$x_t \varphi \equiv \begin{cases} x^{2n} & \text{if } t \leqq k, \\ y^{2n} & \text{if } t > k. \end{cases}$$

The remaining case $\pi = (i\ i+1)$ is dual to $\pi = \iota$.

Lemma 5.5. *If* $\pi \neq \iota$, $\pi \neq (i\ i+1)$ *then*

$$(*) \vdash x_1^{n+2} yz x_2^{n+2} = x_1^{n+2} zy x_2^{n+2}.$$

Proof. If $(*)$ is heterotypical and $(*) \vdash x^{n+2} = x^{n+3}$ then $x_1^{n+2} yz x_2^{n+2} = x_1^{n+2} x_2^{n+2} = x_1^{n+2} zy x_2^{n+2}$. If ($(*)$ is heterotypical and) $(*) \vdash x^{n+1} = x^{n+3}$ then, by (5.10),

$$x_1^{n+2} yz x_2^{n+2} = x_1^{n+2} (yz)^{2n+1} x_2^{n+2} = x_1^{n+2} z^{2n} y^{2n} (yz)^{2n+1} x_2^{n+2} =$$
$$= x_1^{n+2} z^{2n} y^{2n+1} z x_2^{n+2} = x_1^{n+2} z^{2n} y^{2n+1} z^{2n+1} x_2^{n+2}.$$

Put

$$x_t \varphi \equiv \begin{cases} z & \text{if } t = i, \\ y^{2n+1} & \text{if } t = i+1, \\ z^{2n} & \text{else.} \end{cases}$$

Then

$$x_1^{n+2} z^{2n} y^{2n+1} z^{2n+1} x_2^{n+2} = x_1^{n+2} z^{2n+1} \cdot u\varphi \cdot z^{2n} x_2^{n+2} =$$
$$= x_1^{n+2} z^{2n+1} \cdot v\varphi \cdot z^{2n} x_2^{n+2} = x_1^{n+2} z^{2n+1} y^{2n+1} z^{2n} x_2^{n+2}$$

and, by symmetry, $x_1^{n+2} z^{2n+1} y^{2n+1} z^{2n} x_2^{n+2} = x_1^{n+2} zy x_2^{n+2}$.

If $(*)$ is homotypical then $k\pi\neq k$ for some $k\notin\{i,i+1\}$; say, $k<i$. Set

$$x_t\varphi\equiv\begin{cases}x_1^{n+2} & \text{if } t<m_1-1,\\ y & \text{if } t=m_1-1,\\ z & \text{if } t=m_1,\\ x_2^{n+2} & \text{if } t>m_1;\end{cases}\qquad x_t\psi\equiv\begin{cases}x_1^{n+2} & \text{if } t<m_1,\\ yx_2^{n+2} & \text{if } t=m_1,\\ zx_2^{n+2} & \text{if } t=m_1\pi,\\ x_2^{n+2} & \text{else.}\end{cases}$$

We obtain (using also (5.4) and Lemma 5.4 if $m_1\pi=i$)

$$x_1^{n+2}yzx_2^{n+2}=x_1^{n+2}\cdot u\varphi\cdot x_2^{n+2}=x_1^{n+2}\cdot v\varphi\cdot x_2^{n+2}=$$

$$=x_1^{n+2}yx_2^{n+2}z^{e(s)}x_2^{n+2}=x_1^{n+2}yx_2^{n+2}(zx_2^{n+2})^{\varepsilon}=$$

$$=x_1^{n+2}\cdot u\psi\cdot x_2^{n+2}=x_1^{n+2}\cdot v\psi\cdot x_2^{n+2}=x_1^{n+2}(zx_2^{n+2})^{e(m_1)}(yx_2^{n+2})^{e(s)}=x_1^{n+2}zx_2^{n+2}yx_2^{n+2},$$

where $s\pi=m_1$ and

$$\varepsilon=\begin{cases}2 & \text{if } m_1\pi=i,\\ 1 & \text{else.}\end{cases}$$

By symmetry, $x_1^{n+2}zx_2^{n+2}yx_2^{n+2}=x_1^{n+2}zyx_2^{n+2}$.

Corollary 5.6. *Either* $(*)\vdash x_1^{n+2}yzyx_2^{n+2}=x_1^{n+2}y^2zx_2^{n+2}$ *or* $(*)\vdash x_1^{n+2}yzyx_2^{n+2}=$ $=x_1^{n+2}zy^2x_2^{n+2}$.

Proof. If $\pi\neq\iota$, $\pi\neq(i\,i+1)$ then both identities hold by Lemma 4.8. If $\pi=\iota$ then $x_1^{n+2}yzyx_2^{n+2}=x_1^{n+2}y^{e(i)}zx_2^{n+2}=x_1^{n+2}y^2zx_2^{n+2}$ by Lemma 5.1. The case $\pi=$ $=(i\,i+1)$ is dual.

(5.9), (5.10) and Corollary 5.6 imply

Corollary 5.7. *If either the assumptions of Lemma 5.2 or those of Lemma 5.3 are fulfilled then either*

(5.14) $$(*)\vdash x_1\dots x_{n-1}ztzx_ny^{n+2}=x_1\dots x_{n-1}z^2tx_ny^{n+2}$$
or
(5.14′) $$(*)\vdash x_1\dots x_{n-1}ztzx_ny^{n+2}=x_1\dots x_{n-1}tz^2x_ny^{n+2}.$$

Furthermore, applying (5.9), (5.10), (5.14), (5.14′), we obtain

Corollary 5.8. *If either the assumptions of Lemma 5.2 or those of Lemma 5.3 are fulfilled then*

(5.15) $$(*)\vdash wy^{n+2}=w_{(n)}x_{c(1)}\dots x_{c(l)}y^{n+2},$$

$$c(j)\neq c(k)\quad \textit{if}\quad j\neq k,\quad \{x_{c(1)},\dots,x_{c(l)}\}\subseteq X(w)$$

for every $w\in F^n$.

6. Standard forms. Now we are able to construct standard forms for V in the V-ideal J defined by

$$J=\{w\colon \text{there are arbitrarily long terms which equal to } w \text{ in } V\},$$

and for $V(J)$. Although the considerations below could be performed in the same generality as till now, some special cases, in particular the one where $(*)$ is homotypical and $e(k)=2$ only if $k=(i+1)\pi^{-1}$, would demand a separate consideration. Therefore we continue the investigation of $(*)$ accepting in what follows the further restriction

(T) $e(k)=2$ for at least two different values of k.

This suffices for the proof of Theorem A, since the case when $e(k)=2$ for exactly one k has been settled in [6].

Lemma 6.1. *If* $(*)$ *is homotypical,* (T) *holds, and* $|w_j|\geqq n^2-n$ *for* $j=1,2,3$ *then* $w\equiv w_1w_2^2w_3\in J$.

Proof. There is a k such that $e(k)=2$, $k\pi\neq i$. let e.g. $k\pi>i$. First we show that if $|u_j|\geqq n$ for $j=1,2,3$ then there exist $v_1, v_2, v_3\in F$ such that

$$\begin{gathered} u_0\equiv u_1u_2^2u_3=v_1v_2^2v_3\equiv v_0,\quad u_2\equiv u_2'v_2, \\ |v_0|_s>|u_0|_s \quad\text{for}\quad x_s\in X(v_2),\quad |v_j|\geqq|u_j|-n+2\quad\text{for}\quad j=1,2,3. \end{gathered}\tag{6.1}$$

Indeed, put $u_1\equiv u_1'x_{c(1)}\dots x_{c(i-1)}$, $u_2\equiv x_{c(i)}x_{c(i+1)}\dots x_{c(k\pi)}\bar{u}$, $u_3\equiv x_{c(k\pi+1)}\dots x_{c(n)}u_3'$. Then

$$\begin{gathered} u_0\equiv u_1'x_{c(1)}\dots x_{c(i)}(x_{c(i+1)}\dots x_{c(k\pi)}\bar{u})x_{c(i)}x_{c(i+1)}\dots \\ \dots x_{c(k\pi-1)}(x_{c(k\pi)}\bar{u})x_{c(k\pi+1)}\dots x_{c(n)}u_3'=u_1'u_1''(x_{c(k\pi)}\bar{u})^2u_3''u_3' \end{gathered}\tag{$*$}$$

with some $u_1'', u_3''\in F^0$, and $v_1\equiv u_1'u_1''$, $v_2\equiv x_{c(k\pi)}$, $v_3\equiv u_3''u_3'$ meet all requirements of (6.1).

If w is as stated then (6.1) can be applied n times. The term $w=\bar{w}_1\bar{w}_2^2\bar{w}_3$ thus obtained contains every variable of $\bar{w}_2$ at least $n+1$ times. Now suppose some $\tilde{w}\in F$ contains a letter x_s at least $n+1$ times. Then $\tilde{w}$ is of the form $\tilde{w}\equiv u_0\prod_{j=1}^{n+1}x_su_j$ where $u_j\in F^0$ for $j=0,\dots,n+1$. Put

$$x_t\varphi\equiv\begin{cases} u_{t-1}x_s & \text{if}\quad t<i,\\ x_s & \text{if}\quad t=i,\\ u_ix_su_{i+1} & \text{if}\quad t=i+1,\\ x_su_t & \text{if}\quad t>i+1.\end{cases}$$

Then $\tilde{w}\equiv u\varphi$, and $|v\varphi|>|\tilde{w}|$, $|v\varphi|_s>|w|_s$, i.e. we can obtain from $\tilde{w}$ a longer word of the same form, and therefore $\tilde{w}\in J$. The same holds, then, for $\bar{w}$ and hence also for w.

Lemma 6.2. *Suppose* (T) *holds. If* (∗) *is heterotypical then*

$$J = \{w\colon\ u \lhd w \ \text{or}\ v \lhd w\}. \tag{6.2}$$

If (∗) *is homotypical and* $e(k)=2$ *for some* $k \notin \{i\pi^{-1}, (i+1)\pi^{-1}\}$ *then* J *contains the subset*

$$L = \{w\colon\ w \equiv w_{(2n^2-n)}\hat{w}w^{(2n^2-n)},\ u \lhd \hat{w}\}.$$

If (∗) *is homotypical,* $e(k)=2$ *iff* $k\pi \in \{i, i+1\}$, *but* v *is not of the form* (4.1), *then* J *contains the subset*

$$L' = \{w\colon\ w \equiv w_{(n^2+n)}\hat{w}w^{(n^2+n)},\ w_i \notin T(xyx) \cup T(x^2) \cup X$$

$$\text{for some irreducible factor } w_i \text{ of } w\} \cup$$

$$\cup \{w\colon\ w \equiv w'x_cx_dx_cw''x_rx_sx_rw'''; \ |w'|, |w'''| \geqq n^2+2n,\ |w''| \geqq n-2\}.$$

Proof. The first assertion is trivial. If (∗) is homotypical, $e(k)=2$, $k\pi \notin \{i, i+1\}$, say, $k\pi > i+1$, and $w \in L$ then $w \equiv w_{(2n^2-n)}w' \cdot u\varphi \cdot w''w^{(2n^2-n)}$ with some $\varphi \in \operatorname{End} F$, and we can modify the mapping φ in such a way that $x_t\varphi' \equiv x_t\varphi$ if $t < k\pi$, $|x_{k\pi}\varphi'| = n^2$, $|x_t\varphi'| = 1$ if $t > k\pi$, and $w \equiv w_{(2n^2-n)}w' \cdot u\varphi' \cdot w'''w^{(n^2-n)}$. However, then

$$w = w_{(2n^2-n)}w' \cdot v\varphi' \cdot w''' w^{(n^2-n)} \equiv$$

$$\equiv w_{(2n^2-n)}w'(x_{1\pi}^{e(1)} \dots x_{(k-1)\pi}^{e(k-1)})\varphi' \cdot (x_{k\pi}\varphi')^2 \cdot (x_{(k+1)\pi}^{e(k+1)} \dots x_{n\pi}^{e(n)})\varphi' w''' w^{(n^2-n)},$$

and the assumptions of Lemma 6.1 are fulfilled.

Now let $k\pi = i$, $k'\pi = i+1$, $e(k) = e(k') = 2$, $e(j) \leqq 1$ if $j \neq k, k'$. Suppose first that w is contained in L'_1, the first component of L'.

a) If $w \equiv w_{(n^2-n)}\tilde{w}w^{(n^2-n)}$, $|\tilde{w}|_c > 2$ for some $c \in \mathbf{P}$ (in particular, if $|\hat{w}|_c > 2$), then n consecutive applications of (∗) with substituting each time $\varphi_j\colon x_i \mapsto x_c$, $\varphi_j\colon x_{i+1} \mapsto b_j$, $|b_j|_c > 0$ gives us a word w^* with $|w^*|_c > n+1$, and the second part of the proof of Lemma 6.1 verifies the assertion.

b) If $w \equiv w_{(n^2)}\tilde{w}w^{(n^2)}$, $xyxy \lhd \tilde{w}$ (in particular, if $xyxy \lhd \hat{w}$), then $w \equiv$ $\equiv w'x_cax_caw''$, and choosing $\varphi_2 \in \operatorname{End} F$ such that $x_i\varphi \equiv a$, $x_{i+1}\varphi \equiv x_c$, $w\varphi \equiv$ $\equiv w_{(n^2-n)}\bar{w} \cdot u\varphi \cdot \bar{\bar{w}}w^{(n^2-n)}$, we have $|\bar{w} \cdot v\varphi \cdot \bar{\bar{w}}|_c > 2$, and this case can be reduced to a).

c) If $xyzx \lhd \hat{w}$, then $w \equiv w'x_rbx_rw''$, $|b| \geqq 2$. Putting $x_i\varphi \equiv x_r$, $x_{i+1}\varphi \equiv b$, $w \equiv \bar{w} \cdot u\varphi \cdot \bar{\bar{w}}$ with $|\bar{w}|$, $|\bar{\bar{w}}| \geqq n^2$, we have $(*) \vdash \bar{w} \cdot v\varphi \cdot \bar{\bar{w}}$, which gives us case b) as b^2 is a subword of $v\varphi$.

It is easy to see that all possibilities are exhausted by a)—c). Finally, let $w \in L'_2$. By assumption, $\pi \neq \iota$, $\pi \neq (i\ i+1)$, whence there is an $l \neq i$, $i+1$ such that $l\pi \neq l$; let e.g. $l > i+1$ and maximal (then, clearly, $l\pi < l$). Set $w \equiv w_1 \cdot u\psi \cdot w_2 \cdot u\chi \cdot w_3$,

$|w_1|$, $|w_3|\geqq n^2+n$, $x_i\psi\equiv x_c$, $x_{i+1}\psi\equiv x_d$, $x_i\chi\equiv x_r$, $x_{i+1}\chi\equiv x_s$, and

$$x_t\vartheta \equiv \begin{cases} x_t\psi & \text{if } t<l, \\ (x_l\ldots x_n)\psi\cdot w_2\cdot(x_1\ldots x_{i+1})\chi & \text{if } t=l, \\ (x_i x_{i+2}\ldots x_{l+1})\chi & \text{if } t=l+1, \\ x_t\chi & \text{if } t>l+1. \end{cases}$$

We have

$$w\equiv w_1\cdot u\vartheta\cdot w_3 = w_1\cdot v\vartheta\cdot w_3 \equiv ax_r bx_r a'\in L_1',$$

because $|a|\geqq|w_1|\geqq n^2+n$, $|a'|\geqq|w_3|\geqq n^2+n$, $|b|\geqq|x_s\cdot x_{(l\pi^{-1}+1)\pi}\vartheta|\geqq 2$. This completes the proof.

Lemma 6.2 and Proposition 3.10 immediately yield.

Proposition 6.3. *If (T) holds and v is not of the form (**) then $SG(J)$ is h.f.b.*

Proof. Set $\mathrm{T}=\bigcup_{l=0}^{n-1} T(x_1^2\ldots x_l^2)$, and let $I(\mathrm{T})$ be the set of all $w\in F^0$ every semiirreducible factor of which is contained in T. Put

(6.3₁) $$M_1=\{w\colon w_{(n)}\hat{w}w^{(n)},\ \hat{w}\in I(\mathrm{T})\}\cup\{w\colon |w|<2n\},$$

(6.3₂) $$M_2=\{w\colon w_{(2n^2)}\hat{w}w^{(2n^2)},\ \hat{w}\in I(\mathrm{T})\}\cup\{w\colon |w|<4n^2\},$$

(6.3₃) $$M_3=\{w\colon w_{(n^2+n)}\bar{w}\tilde{w}\bar{w}w^{(n^2+n)},\ |\bar{w}|\leqq n,\ \hat{w}\equiv\bar{w}y\bar{w}\in I(\mathrm{T})(y\notin X(\bar{w}\bar{w}))\}\cup\{w\colon |w|<2n^2+2n\}.$$

Lemma 6.2 implies that $F\setminus J\subseteq M_j$ $(j=1,2,3)$, if either $(*)$ is heterotypical or $(*)$ is homotypical and $e(k)=2$ for some $k\notin\{i\pi^{-1},(i+1)\pi^{-1}\}$ or $(*)$ is homotypical, $e(k)=2$ iff $k\pi\in\{i,i+1\}$, and $\pi\neq\iota$, $\pi\neq(i\,i+1)$, respectively. Indeed, if $w\notin M_j$ then w is long enough to be written in the form given in the first bracket of (6.3_j) but with $\hat{w}$ having a semiirreducible factor $w_j\notin I(\mathrm{T})$. Then either $x_ix_{i+1}x_i\lhd\hat{w}$ or $x_1^2\ldots x_n^2\lhd\hat{w}$ and, consequently, if w' is a suffix of length n of $w_{(n)}$, $w_{(2n^2)}$, or $w_{(n^2+n)}$, resp., and, similarly, w'' a prefix of length n of $w^{(n)}$, $w^{(2n^2)}$, or $w^{(n^2+n)}$, we have $u\lhd w'\hat{w}w''$ or $v\lhd w'ww''$ (even $v\lhd\hat{w}$). As the latter case can be reduced to the first one, both imply $w\in J$ by Lemma 6.2. Hence the assertion of the proposition follows by Proposition 3.10, since $M_1\subseteq F_{(2n)}^{[2n]}$, $M_2\subseteq F_{(2n)}^{[4n^2]}$, $M_3\subseteq F_{(2n)}^{[2n^2+2n]}$.

In order to obtain a standard form in J, we prove first

Lemma 6.4. *If (T) holds and $w\in J$ then there is a $\bar{w}\in J$ such that*

$$(*)\vdash w=\bar{w}\equiv w_1x_c^{n+2}w_2,\quad x_c\in X.$$

Proof. If $(*)$ is heterotypical the assertion is obvious; so let $(*)$ be homotypical. There is a $k\leqq n$ which satisfies $e(k)=2$, $k\pi\neq i+1$. We are going to show that for every triple r, s, l of natural numbers there is an $x_c \in X$ and a term $\bar{w}$ such that $w = \bar{w} \equiv w_0 \prod_{j=1}^{s} x_c^r w_j$, $|w_j| \geqq l$ for $j=0, \ldots, s$; this is somewhat more than stated. First consider the case $r=1$. Let $|X(w)|=\lambda$. As $w=w'$ implies now $X(w)=X(w')$, we only have to take a sufficiently long word $\bar{w}$, say, $|\bar{w}| > 2l+\lambda(s-1)(l+1)$, in order to obtain a factorization of the required form. Now suppose the assertion holds for some $r>0$ and arbitrary s, l. In the same way as above, one can find a $w'=w$ such that the same subword $x_d x_c^r x_f$ occurs in w' a sufficiently large number of times and at a sufficiently large distance from each other (this is necessary in order to cover the cases where $k\pi=i-1$ or $k\pi=i+2$; c, d, f need not be different). Say, $w' \equiv \equiv w_0 \prod_{j=1}^{3s} x_d x_c^r x_f w_j$, $|w_j| \geqq n+l$. One can define endomorphisms $\varphi_1, \ldots, \varphi_s$ such that $x_{k\pi}\varphi_j \equiv x_c^r$ for $j=1, \ldots, s$ and $w' \equiv w_0' \prod_{j=1}^{s} u\varphi_j \cdot w_j'$, $|w_j'| \geqq l$. Applying $(*)$, we have $w' = w_0' \prod_{j=1}^{s} v_j \cdot w_j' \equiv w_0'' \prod_{j=1}^{s} x_c^{2r} w_j''$ with some w_j'', $|w_j''| \geqq |w_j'|$. This completes the proof.

Proposition 6.5. *If* (T) *holds and* v *is not of the form* (4.1) *then* $[V(J), V]$ *is finitely based.*

Proof. Let $w \in J$; we can suppose that $w \equiv w' x_c^{n+2} w''$ by Lemma 6.4. First let the assumptions of both Lemma 5.2 and its dual (or those of Lemma 5.3 and its dual) be fulfilled. In virtue of (5.9) (or (5.11)) and its dual, and by Corollary 5.8, we have

$$w = w_{(n-2)} x_d^{n+2} x_{c(1)} \ldots x_{c(l)} x_f^{n+2} w^{(n-2)} \equiv w^* \tag{6.4}$$

$$\big(c(j) \neq c(k) \text{ if } j \neq k\big).$$

Thus, $w^* \in F_{(1)}^{[4n]}$, and the assertion follows by Theorem 3.6, because (i) automatically holds if $(*)$ is homotypical, and if it is not, then, starting from an arbitrary $w'=w$ with a minimal number of variables in $X(w')$, we can transform it to the form (6.4) by steps of the following two types: 1) first, the insertion of the $(n+2)^{\text{nd}}$ power of some variable — however, doing so it is not necessary to introduce new variables, and 2) a sequence of applications of (5.10), (5.11) and (5.14) or (5.14'); besides, (5.10) is always used for reducing the power of the elements, and the other three transformations do not change the set of the variables involved.

For the rest we can suppose, according to the Remark made on p. 321, that the assumptions dual to those of Lemma 5.2 or 5.3 hold, but the assumptions of these lemmata do not, i.e., α) $v \equiv x_1 \ldots x_{i-1} x_i^2 v'$ if $(*)$ is heterotypical, and β) $v \equiv x_1 \ldots \ldots x_{i-1} x_{i\pi}^2 v'$ or γ) $v \equiv x_1 \ldots x_i x_{i+1}^2 v'$ if $(*)$ is homotypical. Let $w \equiv w_{(i-1)} \bar{w} x_{c(0)}^{n+2} x_{c(1)} \ldots \ldots x_{c(l)} w^{(n-2)}$, and write $\bar{w}$ in the form

$$\bar{w} \equiv x_{d(1)}^{\gamma(1)} \ldots x_{d(p)}^{\gamma(p)}, \quad d(i) \neq d(i+1).$$

Suppose that w is chosen in such a way (from the class of all $w'=w$ of the same form), that the vector $\mathbf{v}=\mathbf{v}(w)=(|X(\bar{w})|, p, l)$ is minimal in the lexicographical ordering. We claim that

$$X(\bar{w})\cap\{x_{c(0)}, \dots, x_{c(l-1)}\}=\emptyset. \tag{6.5}$$

Indeed, let $d(q)=c(r)$, $r<l$, and set

$$w_{(i-1)}x_{d(1)}^{\gamma(1)}\dots x_{d(q-1)}^{\gamma(q-1)}\equiv w_1,\quad x_{d(q+1)}^{\gamma(q+1)}\dots x_{d(p)}^{\gamma(p)}\equiv w_2,$$
$$x_{c(1)}\dots x_{c(r-1)}\equiv w_3,\quad x_{c(r+1)}\dots x_{c(l)}w^{(n-2)}\equiv w_4.$$

If $m_2=i$, put

$$x_t\varphi\equiv\begin{cases}x_{d(q)}^{\gamma(q)} & \text{if } t=i,\\ w_2x_{c(o)}^{n+2}w_3 & \text{if } t=i+1,\\ x_{d(q)}^{2n} & \text{if } t>i+1,\end{cases}$$
$$(x_1\dots x_{i-1})\varphi\equiv w_1.$$

We have by the dual of (5.11),

$$w=w_1x_{d(q)}^{\gamma(q)}w_2x_{c(o)}^{n+2}w_3x_{d(q)}^{2n+1}w_4= \tag{6.6}$$
$$=u\varphi\cdot x_{d(q)}^{2n+1}w_4=v\varphi\cdot x_{d(q)}w_4=\begin{cases}w_1x_{d(q)}^{2\gamma(q)}\tilde{w}x_{d(q)}^{2n+1}v_1 & \text{if } i\pi=i,\\ w_1(w_2x_{c(o)}^{n+2}w_3)^2v_2 & \text{if } i\pi=i+1,\\ w_1x_{d(q)}^{2n}v_3 & \text{if } i\pi>i+1\end{cases}$$

with some $\tilde{w}, v_1, v_2, v_3\in F^0$, and in the first case by iteration also

$$w=w_1x_{d(q)}^{2n}v_4.$$

We have a contradiction with the choice of w in all cases.

If $m_2=i+1$, set

$$x_t\varphi'\equiv\begin{cases}x_t\varphi & \text{if } t\leqq i+1,\\ x_{c(r+1)} & \text{if } t>i+1.\end{cases}$$

As in this case $(*)\vdash x^{n+2}=x^{n+3}$, we can make use of the dual of (5.9). Hence

$$\begin{aligned}w&=w_1x_{d(q)}^{\gamma(q)}w_2x_{c(o)}^{n+2}w_3x_{d(q)}^{\gamma(q)}x_{c(r+1)}^{n+2}x_{c(r+2)}\dots x_{c(l)}w^{(n-2)}=\\ &=u\varphi\cdot x_{c(r+1)}^{n+2}x_{c(r+2)}\dots x_{c(l)}w^{(n-2)}=\\ &=v\varphi\cdot x_{c(r+1)}^{n+2}x_{c(r+2)}\dots x_{c(l)}w^{(n-2)}=\\ &=w_1x_{d(q)}^{\gamma(q)}(w_2x_{c(o)}^{n+2}w_3)^2x_{c(r+1)}^{n+2}x_{c(r+2)}\dots x_{c(l)}w^{(n-2)}=\\ &=w_1x_{d(q)}^{\gamma(q)}\cdot w_2x_{c(o)}^{n+2}\cdot x_{c(o)}^{n+2}w_3\cdot w_2x_{c(o)}^{n+2}\cdot w_3^{n+2}x_{c(r+1)}\dots x_{c(l)}w^{(n-2)}=\\ &=w_1x_{d(q)}^{\gamma(q)}w_2x_{c(o)}^{n+2}\cdot(x_{c(o)}^{n+2}w_3)^2w_3^{n+2}x_{c(r+1)}\dots x_{c(l)}w^{(n-2)}=\\ &=w_{(i-1)}\bar{w}x_{c(o)}^{n+2}w_3x_{c(r+1)}\dots x_{c(l)}w^{(n-2)}=\\ &=w_{(i-1)}\bar{w}x_{c(o)}^{n+2}x_{c(1)}\dots x_{c(r-1)}x_{c(r+1)}\dots x_{c(l)}w^{(n-2)},\end{aligned}\tag{6.7}$$

if $r>0$ and

$$(6.8)\qquad w\equiv w_1x_{d(q)}^{\gamma(q)}w_2x_{d(q)}^{n+2}w_4=w_1x_{d(q)}^{\gamma(q)}w_2^2x_{d(q)}^{n+2}w_4=\ldots=w_1x_{d(q)}^{\gamma(q)}w_2^{n+2}x_{d(q)}^{n+2}w_4$$

if $r=0$. In the first case we have reduced l by 1; in the second, $w_1x_{d(q)}^{\gamma(q)}w_2^{n+2}\in J$ and we can create the $(n+2)^{\text{nd}}$ power of a variable in this part of the word by Lemma 5.10. This proves (6.5).

Furthermore, $w_{(i-1)}\bar{w}\notin J$ by the same lemma and (6.5). Let $w_{(l-1)}\bar{w}$ be the longest one of all words it equals to. Then $u\prec\!\!\!\!\mid w_{(l-1)}\bar{w}$ (else we could replace its subword of the form $u\psi$ by the longer word $v\psi$), and $x_1^2\ldots x_m^2x_{m+1}\ldots x_{m+n-i-1}\prec\!\!\!\!\mid w_{(i-1)}\bar{w}$. Indeed, if $e(k)=2$, $k\pi>i+1$ for some k then let

$$x_j\varphi\equiv w_j\equiv y_j\bar{w}_jz_j,\quad x_j\psi\equiv w_j'\equiv\begin{cases}w_j & \text{for } j<k\pi,\\ w_{k\pi}y_{k\pi+1} & \text{for } j=k\pi,\\ \bar{w}_jz_jy_{j+1} & \text{for } j>k\pi\end{cases}$$

($y_j\equiv z_j$ if $|w_j|=1$). We have even

$$v\varphi\cdot y_{m+1}\equiv\Big(\prod_{j=1}^m w_{j\pi}^{e(j)}\Big)y_{m+1}=u\varphi\cdot y_{m+1}\equiv w_1\ldots w_iw_{i+1}w_i\ldots w_ny_{m+1}\equiv$$
$$\equiv w_1'\ldots w_i'w_{i+1}'w_i'\ldots w_n'\equiv u\psi=v\psi\equiv w_{1\pi}'^{e(1)}\ldots w_{m\pi}'^{e(m)},$$

and $|v\psi|=|v\varphi|+2>|v\varphi\cdot y_{m+1}|$. If, on the other hand, $e(k)=2$ iff $k\pi\in\{i,i+1\}$ (this occurs only in case β) with $i\pi\in\{i,i+1\}$), then $\pi\neq\iota$, $\pi\neq(i\,i+1)$, whence $l\pi+1\neq(l+1)\pi$ for some $l\pi>i+1$. Put

$$x_t\psi\equiv\begin{cases}w_t & \text{if } t\in\{i,\ i+1,\ l\pi\},\\ w_{l\pi}w_{l\pi+1}^2 & \text{if } t=l\pi+1,\\ w_t^2 & \text{else.}\end{cases}$$

We have

$$w_{1\pi}^2\ldots w_{n\pi}^2x_1\ldots x_{n-i-1}=w_1^2\ldots w_{i-1}^2w_iw_{i+1}w_iw_{i+2}^2\ldots w_n^2x_1\ldots x_{n-i-1}\equiv$$
$$\equiv u\psi\cdot x_1\ldots x_{n-i-1}=v\psi\cdot x_1\ldots x_{n-i-1}\equiv w_{1\pi}^2\ldots w_{l\pi}\tilde{w}w_{l\pi}\ldots w_{n\pi}^2x_1\ldots x_{n-i-1}.$$

Applying (*) to this latter word, we obtain some word w^* which contains $\tilde{w}^2$ and all factors w_t^2 which do not enter $\tilde{w}$, as well as $x_1,\ldots,x_{n-i-1}$. Hence $|w^*|>|w_{1\pi}^2\ldots\ldots w_{n\pi}^2x_1\ldots x_{n-i-1}|$.

As in the proof of Lemma 4.14, we can conclude now that

$$(6.9)\qquad\begin{aligned}&\bar{w}\equiv\hat{w}\bar{w}^{(n-i-1)},\quad\hat{w}=!\,w_0\prod_{j=1}^s x_{f(j)}w_j,\quad w_j\in\bigcup_{t=0}^{m-1}T(x_1^2\ldots x_t^2),\\&w^*\equiv w_{(l-1)}\overline{w}w^{(n-1)},\quad\overline{w}=!\,\bar{w}x_{c(o)}^{n+\delta}x_{c(1)}\ldots x_{c(l-1)}\quad(\delta=1\text{ or }2).\end{aligned}$$

Let M be the set of all words of the same form as w^*:

$$M_1 = \{w_1\colon |w_1| = i-1\},$$

$$M_2 = \Big\{\hat{w}\colon \hat{w} =! \, w_0 \prod_{j=1}^{s} x_{f(j)} w_j, \quad w_j \in \bigcup_{t=0}^{m-1} T(x_1^2 \dots x_t^2)\Big\},$$

$$M_3 = \{w_3\colon |w_3| = n-i-1\}, \quad M_4 = T(x^{n+2}) \cup T(x^{n+1}),$$

$$M_5 = F_{(1)}, \quad M_6 = \{w_6\colon |w_6| = n-1\},$$

$$M = \big\{w_1 \dots w_6\colon w_i \in M_i, \ X(w_i) \cap X(w_j) = \emptyset \text{ if } i,j \in \{2,4,5\}, \ i \neq j\big\}.$$

We have seen that M is a standard form for V in J. Thus, by Theorem 2.9 and Lemma 3.5, the proof will be accomplished if we show that conditions (i) of Lemma 2.8 and (iii) of Theorem 2.9 are fulfilled. Now (i) holds — if (∗) is homotypical, by Remark 1 made after Corollary 2.10, and if (∗) is heterotypical, by the fact that in assuming that x_c^{n+2} is a subword of w, we could choose x_c as well from the variables of the original word which was to be transformed, and while transforming w to w^* we never needed to introduce a new variable. Furthermore, let $w \equiv \tilde{w}\bar{w} \in M$, $\varphi \in \operatorname{End} F^0$, $\tilde{w}\varphi$ a prefix of some $w' \in M$, and $|\tilde{w}^{(2m-1)}\varphi| = 2m-1$. By Remark 2 (after Corollary 2.10), we can confine ourselves for such φ that

$$|x_t \varphi| = 1 \quad \text{for} \quad x_t \in X(w_1 w_3 w_4 w_6).$$

If $|\tilde{w}| \leqq i-1$, (iii) holds obviously. If $\tilde{w} \equiv w_1 w_2 \bar{v}$, then $w\varphi \equiv u_1 u_2 u_3 u_4 \cdot w'$, $u_i \in M_i$, and we obtain $(w\varphi)^*$ by applying (several times) the dual of either 5.9 or 5.10, and that of 5.14 or of 5.14′, not changing thereby $\tilde{w}\varphi$. Finally, if $\tilde{w} \equiv w_{(i-1)} \tilde{w}_2 \hat{v}$, $\tilde{w}_2 \hat{v}$ a prefix of $\hat{w}$, and either $\hat{v} \equiv x_{f(j)} x_{d(1)}^2 \dots x_{d(t-1)}^2 x_{d(t)}$, $\lambda \leqq 2$, $|w_j| \geqq 2t$; then, by assumption, $|x_{f(j)}\varphi| = |x_{d(1)}\varphi| = \dots = |x_{d(t)}\varphi| = 1$, and $X(\tilde{w}_2\varphi) \cap X(\bar{w}\varphi) \subseteq X(w_6\varphi)$. Thus, we can transform $(w_{(i-1)} w_2 x_{f(j)}) \varphi^{(i-1)} (x_{d(1)}^2 \dots x_{d(t)} \bar{w})$ to a standard form $\overline{\overline{w}}$ without changing either w_0 or w_6; thereby we "standardize" the whole word $w\varphi$, without changing $(w_{(i-1)} \tilde{w}_2 x_{f(j)})\varphi$. Or $\tilde{w}_2 \equiv \emptyset$, $|\hat{v}| = 2m-1$, and the assertion is obvious once more. This completes the proof.

Theorem A follows now from [6], Proposition 4.1, Lemma 4.2, Propositions 6.3, 6.5 and 2.5.

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SOMOGYI B. U. 7
6720 SZEGED, HUNGARY